ESANN 2026 proceedings, European Symposium on Artificial Neural Networks, Computational Intelligence and Machine Learning. Bruges (Belgium) and online event, 22-24 April 2026, i6doc.com publ., ISBN 9782875870964. Available from http://www.i6doc.com/en/.

Domination Reliability Analysis Based on Graph Features Using Generalized Matrix LVQ

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Abstract. Evaluating domination reliability—a network reliability measure related to service networks—is a computationally expensive task, due to its proven NP-hardness. To address this challenge, we propose an interpretable prototype-based classification approach that predicts domination reliability levels from selected graph features using Generalized Matrix Learning Vector Quantization (GMLVQ) with a particular focus on how these graph features influence the predicted reliability levels. The interpretability is enhanced by a physically motivated visualization of an associated threshold graph, which is derived from the learned relevance matrix.

1 Introduction

Reliability theory is concerned with quantifying the probability that a system continues to operate despite random failures of its interacting components which are taken as vertices of a graph and the interactions are represented by respective edges [1]. Classical network reliability addresses connectivity properties; in contrast, *domination reliability* considers whether the set of operating vertices of a graph dominates the entire graph. The graph can be viewed as an abstract representation of a service network, where each vertex supplies an identical service to its neighbors as well as to itself. The domination reliability of such a network is the probability that the failed vertices get served its operating neighbors.

This measure was introduced and analyzed in [4], where its computational complexity was shown to be NP-hard, implying that exact computation is feasible only for relatively small graphs. Its significance extends well beyond service networks and lies in the fact that the reliability of any coherent system can be expressed as the domination reliability of an associated graph [4].

In recent years, machine-learning techniques, such as artificial neural networks, support vector machines, and graph neural networks, have gained attention in reliability analysis [7, 10]. These techniques generally do not provide interpretable results, which would be valuable for optimizing network reliability. Specifically to the domain of domination reliability, no machine-learning method has yet been applied. We aim to contribute to this in the present work.

Our methodology differs from existing machine-learning approaches for other reliability measures by achieving interpretable reliability estimates by learning reliability levels from relevant graph features. More precisely, we apply Generalized Matrix Learning Vector Quantization (GLMVQ)—a prototype-based classification scheme—that, based on structural graph features, identifies prototypes in the feature space corresponding to reliability levels. This enables an interpretation of the contribution of the graph features to the classification, which is visualized by an associated threshold graph on the feature set. Such an

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interpretation is valuable in its own right, independently of prediction, because it provides insight into structural properties underlying domination in graphs.

This paper is structured as follows. Section 2 recalls the definition of domination reliability and introduces the graph features used for learning. Section 3 outlines the GMLVQ approach in this setting. Section 4 presents the experimental design and the achieved results. Section 5 concludes with a discussion and outlook.

2 Domination reliability and relevant graph features

Let $G = (V, E)$ be a finite simple undirected graph having vertex-set V and edge-set E. Each vertex $v \in V$ operates independently with probability ρ (the so-called *vertex reliability*) and fails with probability $1 - \rho$, while the edges are assumed to be perfectly reliable. The set of operating vertices is denoted by $X \subseteq V$. The graph continues to satisfy its operational criterion if X contains a dominating set, that is, if every vertex in $V \setminus X$ is adjacent to at least one vertex in X. The *domination reliability* of G with respect to ρ is defined as the probability that this event occurs and is denoted by $\mathrm{DRel}(G, \rho)$ [4]. As shown in [4], computing $\mathrm{DRel}(G, \rho)$ is NP-hard in general, and efficient algorithms are currently known only for cographs.

As further shown in [4], for graphs of small order the inclusion–exclusion principle can be applied to express the domination reliability as

$$\mathrm{DRel}(G, \rho) = \sum_{J \subseteq V} (-1)^{|J|} (1 - \rho)^{|N_G[J]|}. \tag{1}$$

In this equation, $N_G[J]$ denotes the closed neighborhood of J in G, consisting of all vertices that are either in J or adjacent to at least one vertex in J. Unfortunately, this expression grows exponentially with $|V|$ and can therefore be applied only to graphs of moderate size (typically up to about 20 vertices).

Since exact efficient methods are out of reach due to the proven NP-hardness of the problem, our goal is to predict domination reliability in terms of discrete reliability levels. To this end, we generate random graphs, for which the exact domination reliability can still be computed using Eq. (1), and from each graph we extract structural features suitable for learning meaningful reliability levels. The resulting model can then be applied to graphs of different orders, providing predictions of their reliability levels and revealing how the extracted features (and their interactions) influence these learned reliability levels.

The selection of structural features for classification is guided by three considerations: First, the number and scale of the features should be fixed, that is, independent of the number of vertices. This ensures that the learned model can be applied to graphs of different orders and prevents a full specification of the graph (e.g., via its adjacency or incidence matrix). Second, the features should be correlated with domination in graphs, meaning that they should capture aspects of the neighborhood structure. Third, the features should be computationally inexpensive to compute; NP-hard quantities are therefore excluded. The features satisfying these criteria include statistics derived from vertex degrees, common-neighbor counts, as well as spectral and clustering characteristics.

The feature vector associated with each graph G is a 16-dimensional vector $\mathbf{t} = (t_1, t_2, \ldots, t_{16}) \in [0, 1]^{16}$, where $t_1, \ldots, t_{16}$ are normalized versions of the

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following quantities (in this order): the graph's density; the maximum, average and standard deviation of its degree sequence; the average distance; the diameter; the average eccentricity; the average core number; the spectral radius; the algebraic connectivity; the average clustering coefficient; the average number of common neighbors over all vertex pairs, over all adjacent vertex pairs, and over all non-adjacent vertex pairs; the maximum number of common neighbors over all vertex pairs; and finally the homogeneous vertex reliability ρ.

By normalization we mean that each value is divided by the maximum possible value for any simple graph on n vertices (for example, by $n-1$ in the case of the maximum degree, or by $(n+1)/3$ in the case of the average distance), ensuring that all features lie within the unit interval. It is worth noting that the feature vector has a fixed length that is independent of the number of vertices.

Some of the graph features are meaningful only for connected graphs. This is not a limitation, since the domination reliability of a disconnected graph can be computed from the domination reliabilities of its connected components. For connected graphs, these features are easy to compute and provide a compact representation of the graph's structural properties that are relevant to the existence of dominating sets. The central question is whether they encode sufficient information to enable the prediction of domination reliability levels.

3 Generalized Matrix LVQ

Generalized Matrix Learning Vector Quantization (GMLVQ) [9] is a prototype-based classification method combined with adaptive metric learning for the underlying nearest-prototype-paradigm (NPP). GMLVQ optimizes the localization of class dependent prototypes for class detection in parallel with metric optimization for the NPP. GMLVQ without metric adaptation is denoted as GLVQ.

Let $\mathcal{X} = \{\mathbf{x}_j \in \mathbb{R}^n,\, j = 1, \ldots, m\}$ be a set of training data with class labels $c(\mathbf{x}_j) \in \mathcal{C} = \{1, \ldots, C\}$, and let $\mathcal{W} = \{\boldsymbol{w}_k \in \mathbb{R}^n,\, k = 1, \ldots, l\}$ be a set of prototypes with predefined class responsibilities $c(\mathbf{w}_k) \in \mathcal{C} = \{1, \ldots, C\}$ such that each class is represented by at least one prototype. Classification of an unknown sample $\mathbf{x}$ follows a winner-takes-all rule realizing the NPP: $c(\mathbf{x}) = c(\mathbf{w}_s)$ with $\mathbf{w}_s = \arg\min_{\mathbf{w}_k \in \mathcal{W}} d(\mathbf{x}, \mathbf{w}_k)$, where d is a dissimilarity measure. The cost function to be minimized in standard GLVQ for the training data is defined as $E(\mathcal{X}, \mathcal{W}) = \sum_{\mathbf{x}_j \in \mathcal{X}} f(\mu(\mathbf{x}_j))$, where f is a monotonically increasing function and

$$\mu(\mathbf{x}) = \frac{d(\mathbf{x}_j, \mathbf{w}^+) - d(\mathbf{x}_j, \mathbf{w}^-)}{d(\mathbf{x}_j, \mathbf{w}^+) + d(\mathbf{x}_j, \mathbf{w}^-)} \in [-1, 1]\,.$$

μ is the classifier function based on the dissimilarity measures $d(\mathbf{x}_j, \mathbf{w}^+)$ and $d(\mathbf{x}_j, \mathbf{w}^-)$ to the closest prototype $\mathbf{w}^+$ with a matching label, and the closest prototype $\mathbf{w}^-$ with a non-matching label, respectively. A correct classification is obtained for $\mu(\mathbf{x}_j) < 0$. Metric learning is again achieved via stochastic gradient descent (SGD) on $E(\mathcal{X}, \mathcal{W})$, which is, in fact, an approximation of the overall classification error. For d being a squared seminorm or semimetric dissimilarity, this SGD implicitly maximizes the hypothesis margin, resulting in a robust and interpretable classifier [3, 8].

Unlike GLVQ, GMLVQ improves classification performance by incorporating adaptive metric learning, also known as relevance learning. For this purpose the parametrized quadratic form $d_\Lambda(\mathbf{x}_j, \mathbf{w}_k) = (\mathbf{x}_j - \mathbf{w}_k)^\top \Omega^\top \Omega (\mathbf{x} - \mathbf{w}_k)$, where

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$\Omega \in \mathbb{R}^{n \times n}$ and $\Lambda = \Omega^\top \Omega$, is the so-called *relevance matrix* [9]. Metric adaptation is achieved by application of an SGD-scheme to the matrix entries Ω_{ij}. The diagonal elements of the learned relevance matrix Λ can be interpreted as feature relevances, as they emphasize individual feature dimensions in the resulting classification; this is referred to as the *classification relevance profile*. The off-diagonal elements, in contrast, capture possible interdependencies among the features. Generalized Relevance Learning Vector Quantization (GRLVQ) is like GMLVQ, but with the relevance matrix restricted to the diagonal, thereby focusing on individual feature weights and discarding cross-feature interactions [6].

Notably, adaptive metric learning can improve classification accuracy while also providing valuable interpretability [2]. In the following section, we demonstrate this in the context of domination reliability analysis.

4 Experimental Design and Results

We constructed two data sets $\mathcal{D}_1$ and $\mathcal{D}_2$, each containing $N = 500$ samples of the form $(\mathbf{t}, \tau)$, where $\mathbf{t}$ is a graph feature vector, as described in Section 2, and τ the graph's domination reliability. The samples are prepended by the Graph6 string of the graph and its order, which are not used in the learning procedure but are included for reference purposes. The data sets and the code for generating the data sets are publicly available[1].

Since the domination reliability can be computed in reasonable time for graphs with up to roughly 20 vertices, this graph size provides a good experimental framework. The graphs were generated from the Erdős-Rényi random graph model $G(n, p)$ for $n = 10, \ldots, 20$ and $p = \frac{1}{2}\ln(n)/n$, ensuring a moderate edge density, which mimics the structural scale in many real-world networks, where the average degree stays bounded even as the network grows. Disconnected graphs were discarded for reasons explained in Section 2. In data set $\mathcal{D}_1$, the homogeneous vertex reliability ρ of each graph, corresponding to the last entry in the graph's feature vector, was chosen uniformly at random from the intervall $[0.85, 0.95]$, whereas in $\mathcal{D}_2$, it was fixed to 0.9. Hence, in both data sets, the homogeneous vertex reliabilities are around 0.9, a value commonly regarded as acceptable in practical settings. For each graph, the domination reliability was computed from Eq. (1), based on its homogeneous vertex reliability ρ.

Special quantiles c_α of τ are used to define the reliability levels, and thus the labels $c(\mathbf{t})$. More precisely, we consider the set $\mathcal{C} = \{c_{0.25}, c_{0.5}, c_{0.75}\}$ of class labels, representing the reliability levels *unreliable*, *reliable*, and *highly reliable* in terms of quantiles. These quantiles partition the interval $[0, 1]$ equidistantly.

In order to rescale the data and bring the features to the same order of magnitude, a z-score transformation was applied as a preprocessing step.

We observed that the data are not linearly separable, necessitating the use of three prototypes per class in GMLVQ and GRLVQ to adequately represent the underlying data structure. Further, we applied a 3-fold cross-validation. The resulting averaged accuracies for training and testing, $\mathsf{acc}_{\mathsf{train}}$ and $\mathsf{acc}_{\mathsf{test}}$, respectively, along with their standard deviations, are presented in Table 1, which reveals that GMLVQ achieves higher training and test accuracies compared to GRLVQ on both datasets. This demonstrates the advantage of learning full relevance matrices. Comparing the datasets, GMLVQ predicts the reliability levels

[1] `https://git.hs-mittweida.de/domrel/domrel-ml`

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Table 1: Averaged training and test accuracies with their standard deviations for GMLVQ and GRLVQ.

		$\mathcal{D}_1$	$\mathcal{D}_2$
GMLVQ	$\text{acc}_{\text{train}}$	(0.9425 ± 0.0090)	(0.9042 ± 0.0138)
	acc_{test}	(0.8967 ± 0.0115)	(0.8667 ± 0.0115)
GRLVQ	$\text{acc}_{\text{train}}$	(0.8317 ± 0.008)	(0.8042 ± 0.0118)
	acc_{test}	(0.7933 ± 0.0231)	(0.800 ± 0.0458)

better on $\mathcal{D}_1$ than on $\mathcal{D}_2$. This indicates that $\mathcal{D}_1$ exhibits a more distinctive feature structure relative to the classes, whereas $\mathcal{D}_2$ appears more challenging, leading to slightly reduced performance despite robust generalization.

To enhance the interpretability of the learned relevance matrix, for any threshold δ we consider a graph $G(\delta)$, whose vertices correspond to the features selected in Section 2 and whose edges represent pairs of distinct vertices i and j such that $\Lambda_{i,j} > \delta$. This graph is visualized by a force-directed graph drawing algorithm [5], which uses attractive forces between adjacent vertices and repulsive forces between distant vertices. In this visualization, centrally positioned vertices (features) have a particularly strong influence on the classification, and pairs of vertices (combinations of features) that are located close to each other exert a greater joint influence on the classification than those that are far apart.

Figure 1 shows $G(\delta)$, where δ is chosen as the median of all non-diagonal elements of Λ. In view of this graph, features 2, 4, 12, and 16 are most important with respect to classification. These features correspond to the maximum degree, the standard deviation of the degrees, the average common neighbor count over all vertex pairs, and the vertex reliability. We conclude that these degree-related and neighborhood-related features have a particularly strong influence on the classification of reliability levels, and thus on domination reliability, which may prove useful for further graph-theoretical investigations. Notably, the spectral radius (feature 9), given by the largest eigenvalue of the adjacency matrix, has a greater impact on the classification than the algebraic connectivity (feature 10), given by the second-smallest eigenvalue (counting multiplicities) of the Laplacian matrix. The latter has virtually no influence on the classification. This indicates

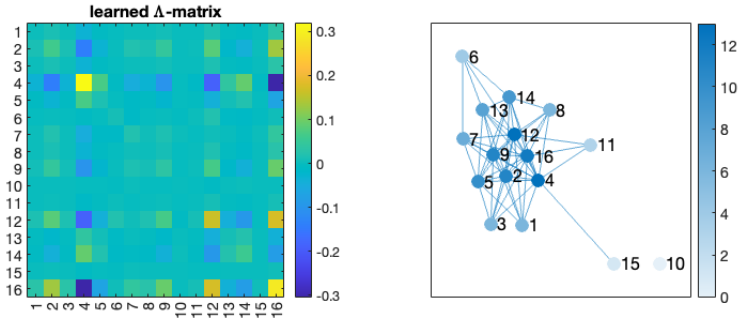

Figure 1: Learned relevance matrix Λ of $\mathcal{D}_1$ and the threshold graph $G(\delta)$

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that, with respect to domination in graphs, the eigenvalues of the adjacency matrix are more significant than those of the Laplacian matrix.

5 Conclusion

The estimation of domination reliability levels from selected graph features using GMLVQ provides an effective alternative to the computationally demanding exact computation of domination reliability. A key outcome of this work is not merely the prediction itself, but the interpretability of the learned relevance matrix. By revealing the relative importance and interactions of the structural features, the learned Λ matrix provides insight into how domination reliability depends on the graph's structural properties. To further enhance interpretability, a graph-based visualization highlights the key structural patterns and feature interactions that are particularly important for the classification of the reliability levels. Overall, the results show that degree- and neighborhood-based features exert the strongest influence on the classification of domination reliability levels.

Future work may address the prediction of other difficult graph characteristics such as the collision probability of random colorings. We furthermore see great potential in advancing the graph-theoretical visualization of the Λ matrix via $G(\delta)$ by incorporating recent developments in the field of graph drawing.

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