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ONCE MORE ON THE INFORMATION CAPACITY OF HOPFIELD NETWORK

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Abstract: This work concerns an investigation of a Hopfield fully connected neural network (NN) with ± 1 neuron states by computer simulation. The results of computer experiments were approximated by formulae which are commonly used in analytical approaches. It allowed us to extrapolate the results of the computer simulation to the case of thermodynamical limit when the number of NN neurons $N \to \infty$. Some results of such extrapolations were compared with those which were obtained analytically earlier. These two kinds of results were found to be close each other. Rest of the results was obtained for the first time. The information amount which can be extracted from the network due to a certain proposed decoding procedure was calculated. It was shown that this procedure allowes one to extract information from the network which amount is close to the well known high limit the network's information capacity $I \approx 0.14N^2$.

1. Introduction

Since the Hopfield paper appeared [1], the Hopfield NN has been one of the most extensively investigated objects in NN theory. Main interests concern its informational properties. The high limit of information amount which can be extracted from the network is defined by the Shannon theorem and cannot exceed the entropy of its connection matrix $H(T)$ which in turn for symmetrical fully connected NN cannot exceed $N(N-1)H(T_{i,j})/2$, where $H(T_{i,j})$ is entropy of connection weight $(i \neq j)$ which is proportional to $\mathbf{ld}(L)$ for a large L, where L is the number of written patterns and $\mathbf{ld}(L)$ is its logarithm at base 2. This formulae implies that the whole number of connection weights which can be supposed to be statistically independent memory elements is $N(N-1)/2$. That is, the entropy of the connection matrix $H(T)$ which gives the high limit of information which can be extracted from a neural network for every decoding procedure may be called its information capacity. But it has been shown [2], [3], [4] that an information amount which is close to $H(T)$ may be extracted from NN only under very special class of decoding procedures which are called complex [2]. Retrieval procedures which correspond to the use of an NN as

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associative memory do not belong to this class so they are called simple [2]. For simple procedures with the Hebbian learning rule, the information amount I which is extracted from the network can not exceed value $I = EN^2$, where E is the coefficient of proportionality which is called effectivity and depends on the type of retrieval procedure. Entropy based estimation of the high limit of E for every simple procedure gives the value 0.72 [3], [4]. Because of this result, the value $I = 0.72N^2$ must be considered as the information capacity of a fully connected Hopfield network with the Hebbian learning rule.

The third high limit of information amount which can be extracted from NN is given by macrodynamic approach [5], [6] and corresponds to abrupt disappearence of a NN "ferromagnetic" properties at $L \approx 0.14N$. Thus E cannot exceed 0.14. But for the most retrieval procedures the value of E may be much less even than this limit.

It is beyond belief, but an evaluation of the information amount which is extracted from the Hopfield network for any retrieval procedure has yet appeared. Here we evaluate this amount for a certain natural decoding procedure described in the following.

2. Model description

In the case of a fully connected Hopfield network, the elements of the connection matrix are defined by the equations

$$T_{i,j} = \frac{1}{N}\sum_{l} X_i^l X_j^l , \quad T_{i,j} = 0 \;\; i,j = 1,\ldots,N, \;\; l = 1,\ldots,L$$

where X^l are bipolar vectors written into the memory. Let these vectors be called etalons. Their elements are assumed to be statistically independent and equal to 1 or -1 with probability $p = 0.5$.

In the case of discrete time, the neural network dynamics is defined by the equation

$$X_i(t+1) = \text{sgn}\,(\sum_{j} T_{i,j} X_j(t)) \quad i,j = 1,\ldots,N$$

where

$$\text{sgn}\,(\eta) = \begin{cases} +1 & \text{if } \eta > 0 \\ \text{unchanged} & \text{if } \eta = 0 \\ -1 & \text{if } \eta < 0 \end{cases}$$

It is known that for asynchronous mode (that means that at each time step only one neuron can change its state) this activity reaches one of the stable states. In this work the sequence of neurons which may change their states is given.

When relatively few etalons are stored, the stable states will correspond to these etalons. But there exists a large number of spurious stable states i.e.

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states which are far from written etalons. The measure of distance between two network states X^1 and X^2 is given by the formula

$$m(X^1, X^2) = \frac{1}{N} \sum_j X_j^1 X_j^2 , \quad j = 1, \ldots, N$$

where m is called overlap and is ranged between -1 and 1.

It is evident that network informational properties depend on

- probability P_1 that given etalon has in its vicinity a stable state
- mean distance between etalons and their nearest stable states m_s
- shape and size of etalons' basins of attractivity
- number of spurious stable states M
- shapes and sizes of spurious stable states basins of attractivity

Most of these properties of the Hopfield NN have already been investigated (see for example [5,6]). The main goal of this work is to evaluate how these properties affect the information which is extracted from the memory under the considered decoding procedure and to find by computer experiments asymptotical equation for E in the case of $N \to \infty$.

3. Decoding procedure

The simplest and most natural decoding procedure which is based on the ability of NN to produce stable states near written etalons with rather large basins of attractivity is the following. Let random NN states X^0 be sequentially taken as its initial states. Let us choose from among them only such states that have overlap $m(X^0, X^s) > m_{th}$ where X^s is a stable state to which NN activity tends from initial state X^0 and m_{th} is the given threshold overlap value. If such stable state is in the vicinity of one of etalons we shall call it true else false. In computer simulation we assumed that stable state X^s is in the vicinity of etalon X^i if $m(X^i, X^s) > 0.9$.

It may be shown [7] that information amount which is extracted from NN due to this procedure is given by the following formulae:

$$I \approx P_1 LN(1 - h((1 + m_s)/2) - \frac{1}{N} \mathrm{ld} \frac{1}{P}) \qquad (1)$$

where $h((1 + m_s)/2) = H(X^i/X^s)$ is conditional entropy of etalone X^i when stable state in its vicinity X^s is known, $h(p) = -p\mathrm{ld}p - (1 - p)\mathrm{ld}(1 - p)$ is Shannon function, $p_s = (1 + m_s)/2$ is probability that state of given neuron in X^s coincides with its state in X^i , P is probability that given stable state choiced by the described above procedure is true.

To calculate P we may assume

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$$P = \frac{LP_1 q_1(m_{th})}{LP_1 q_1(m_{th}) + q_2(m_{th})M} \tag{2}$$

where M is the total number of spurious stable states and $q_1(m_{th})$ and $q_2(m_{th})$ are the probabilities that the initial state which has overlap exceeded m_{th} with given stable state converges to this state.

Thus, evaluation of information amount which is extracted from the network due to considered decoding procedure comes to the evaluation of P_1, m_s, M, $q_1(m_{th})$ and $q_2(m_{th})$ and formulas (4) and (5) show how the main network properties enumerated at the end of the previous section influence its information capacity.

4. Computer simulation

4.1 Evaluation of P_1 and m_s

To evaluate the probability P_1 that a given etalone has a stable state in its vicinity and mean overlap m_s between an etalone and such a stable state, the etalones themselves were used as initial network states. Computer experiments were carried out with 10 different random sets of etalones for the values L and N. Each written etalone was tested for the presence of a stable state in its vicinity. Thus, each value of P_1 was calculated by 10 L trials and each value of m_s was calculated by 10 $P_1 L$ trials.

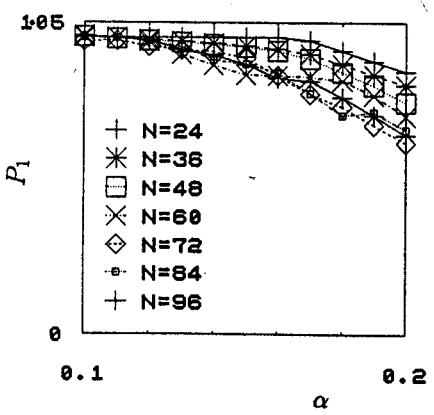

Fig.1.

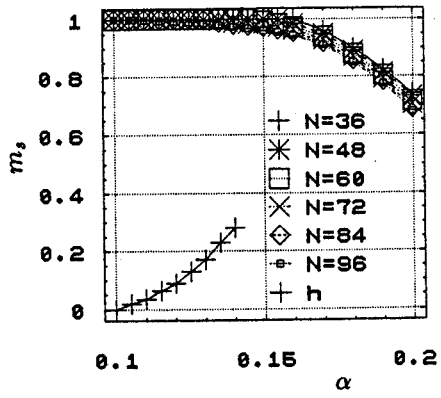

Fig.2.

The obtained dependences of P_1 on $\alpha = L/N$ under the different values of N are shown in Fig.1. Here and in the following values, all considered parameters were calculated by linear interpolation of their values obtained in discrete values of α corresponding to the integer values N and L. This dependence may be approximated by the formulae

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$$P_1 \approx \frac{1}{1 + a_1\sqrt{N}\exp(a_2(\alpha - a_3))} \tag{3}$$

where $a_1 \approx 0.004$, $a_2 \approx 0.5$ and $a_3 \approx 0.14$. For $N \to \infty$ we may assume

$$P_1 = \begin{cases} 1 & \text{for } \alpha < a_3, \\ 0 & \text{for } \alpha > a_3. \end{cases}$$

The obtained critical value of $\alpha = a_3 \approx 0.14$ coincides with the evaluations obtained in many other works.

The obtained dependences of m_s on α for different values of N is shown in Fig.2. It can be seen that for $N \to \infty$, this dependence tends to a limit curve which was used for calculation of $h((1 + m_s(\alpha))/2)$. This dependence is also shown in Fig.2 for $\alpha < 0.14$. It can be seen that this term of formulae (2) is relatively small and may be ignored under calculation of I for $\alpha < 0.1$.

4.2 Evaluation of $q_1(m_{th})$

To evaluate the probability $q_1(m_{th})$ that a given initial state X^0 which has an overlap with a given etalone $X^i m(X^0, X^i) > m_{th}$ converges to a stable state in its vicinity (if it exists) five different random noisy variants of each etalone with given initial overlap $m(X^0, X^i) = m_{th}$ are sequentially used as initial network states. In other respects the experiments were the same as described in the previous section. Thus each value of $q_1(m_{th})$ was calculated by $50P_1L$ trials.

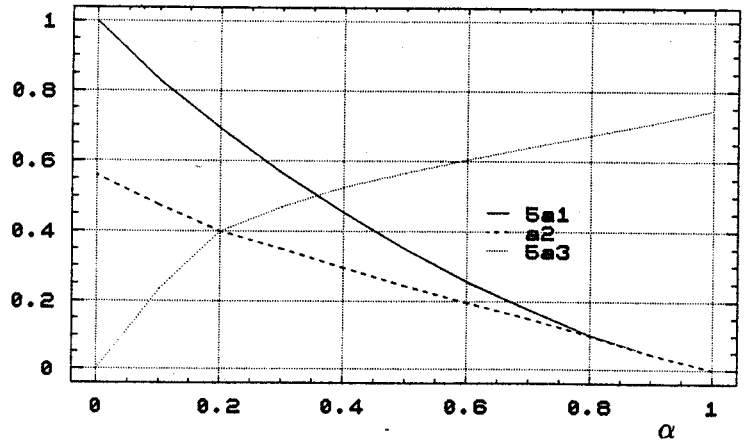

Fig.3.

The obtained dependences of q_1 on α and m_{th} under the different values of N is similar to one shown in Fig.1. These dependences may also be approximated by formula (3) where a_1, a_2 and a_3 are functions of m_{th}. Approximate values of these functions are shown in Fig.3. For $N \to \infty$ we may assume that

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$$q_1 = \begin{cases} 1 & \text{if } \alpha < \alpha_{cr} = a_3(m_{th}) \text{ (or } m_{th} > a_3^{-1}(\alpha)) \\ 0 & \text{in the opposite cases.} \end{cases}$$

The obtained dependence of critical values of $m_{th} = a_3^{-1}(\alpha)$ which determine the border of attractivity basine for different values of α is close to that obtained in the paper [6] by the macrodynamic approximation of retrieval procedure.

4.3 Evaluation of M

To evaluate the whole number of spurious stable states a large number N_r of random network states was used for its initial states. The whole number M' of different stable spurious states which appeared under such testing was calculated. A stable state was assumed to be spurious if its overlaps with all etalones was less than 0.9.

Let P_2 be the probability that a spurious stable state will appear during the a given trial. Then the probability that a given spurious state will appear is P_2/M. The probability P' that a given spurious state will not appear for all N_r trials is

$$P' = (1 - P_2/M)^{N_r} = \exp(-N_r P_2/M)$$

Then the mean number of different spurious states which appeared after N_r trials is

$$M' = M(1 - \exp(-N_r P_2/M))$$

This formula was used for calculating M by M' and P_2 which were averaged by 10 different random sets of etalones for different values of N and L. The dependence of M on α for different values of N may be approximated by the formula

$$M = 0.04N \exp(0.1L) \tag{4}$$

which qualitatively corresponds to the result obtained in the paper [5] by thermodynamic approach.

4.4 Evaluation of $q_2(m_{th})$

To evaluate this dependence and the value of P which is contained in formula (1) and directly determines the information amount obtained from the external source, we fully simulated the procedure described in section 3 for different values of m_{th}. Probability P was calculated by N_r trials for each given set of etalones which then was averaged by 10 different random sets of etalones for each of the values of N and L. Unfortunately, increasing m_{th} causes an exponential decrease of probability that the given trial with X^0 as initial state will provide the stable state X^s such that $m(X^0, X^s) > m_{th}$. That is, the number of trials N_r which is necessary for accurate evaluation of P must exponentially increase. So we succeed in evaluation of P only for $m_{th} \leq 0.4$. Dependence of $I' = (1/N)\mathbf{ld}(1/P)$ on α for different values of N and m_{th} is shown in Fig.4. It is seen that this dependence is approximately linear.

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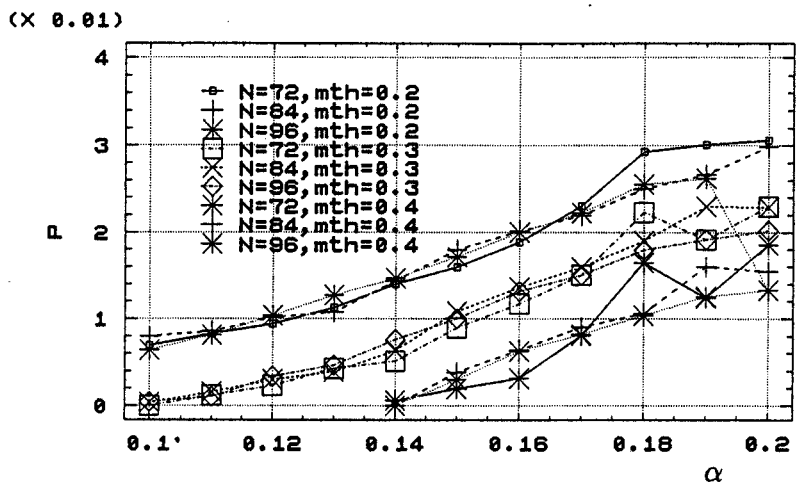

Fig.4.

And for $I^{'} > 0$ it may be approximated by the formula

$$I^{'} \approx 0.2(\alpha - a_3(m_{th}))$$

It is also seen that there exists a range of α and m_{th} where the value of P is close to 1 in spite of the large number of M which is given by formula (4) and is contained in the denominator of formula (3) for P. This means that in this range values of q_2 fall with N increase much faster than values of q_1 . Formula (3) provides the a possibility for evaluating the dependence of q_2 on α, m_{th} and N.

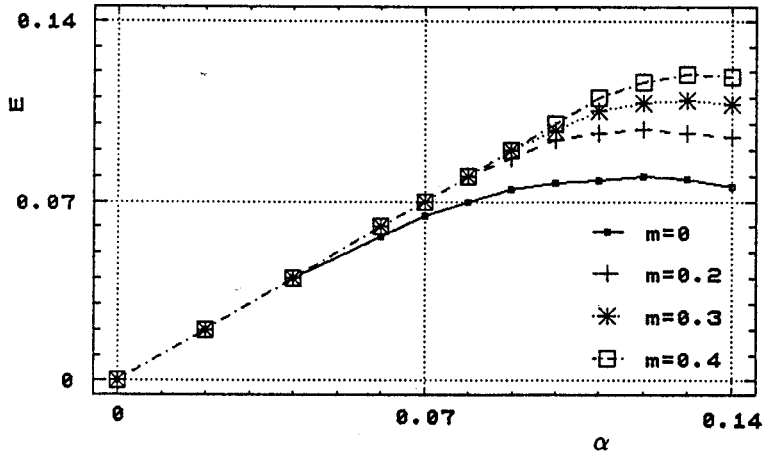

Fig.5

At last we can evaluate the information amount which can be extracted from the network for the described decoding procedure. Dependences of E on α for different values of m_{th} and $N \to \infty$ which were calculated by formula (2) are

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shown in Fig. 5. It can be seen that the maximal value of information which can be extracted from the network with m_{th} increases and reaches a value which is close to the evident high limit $E = 0.14$.

5. Conclusions

Computer simulation of the Hopfield network confirmed the evaluations of its information properties which were obtained earlier by various analytical approximations (see for example Amit et al., 1987, 1988; Amari, Maginu, 1988). The results of computer experiments were approximated by formulae which are commonly used in analytical approaches. It allowed us to extrapolate the results of the computer simulation to the case of thermodynamical limit when the number of NN neurons $N \to \infty$. Some results of such extrapolations were compared with those which were obtained analytically earlier. These two kinds of results were found to be close each other. Rest of the results was obtained for the first time. The information amount which can be extracted from the network due to a certain proposed decoding procedure was calculated. It was shown that this procedure allowes one to extract information from the network which amount is close to the well known high limit the network's information capacity $I \approx 0.14N^2$. Furthermore, it provided an evaluation of the attractivity of true and false stable states. It was shown that there exist a range of α and m_{th} where the attractivity of spurious states falls much faster than the attractivity of true states. This property of the Hopfield network may be used as the basis of a decoding procedure which allows one to approach the natural high limit of $E \approx 0.14$.

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