ESANN'1994 proceedings - European Symposium on Artificial Neural Networks
Brussels (Belgium), 20-21-22 April 1994, D-Facto public., ISBN 2-9600049-1-4, pp. 55-60

Diluted Neural Networks with Binary Couplings : a Replica Symmetry Breaking Calculation of the Storage Capacity

J. Iwanski and J. Schietse

Limburgs Universitair Centrum
Universitaire Campus, B-3590 Diepenbeek, Belgium

Abstract. We perform a one-stage Replica Symmetry Breaking calculation of the storage capacity of the annealed-dilution network with binary couplings and show that a Replica Symmetry calculation with the additional condition that the entropy vanishes at $T = 0$ gives the same result. This generalizes a result previously obtained by Krauth and Mézard for the non-diluted case.

The network is a perceptron with N input and 1 output neurons. Each neuron is a two-state unit described by an Ising variable. The network will function as a memory device for storage of $p = \alpha N$ random patterns or input-output pairs $\{\boldsymbol{\xi}^\mu, \zeta^\mu\}$ $(\mu = 1 \ldots p)$ provided N coupling coefficients J_j can be found such that

$$\zeta^\mu = sgn\left[\sum_{j=1}^{N} J_j \xi_j^\mu\right] \qquad (\mu = 1 \ldots p). \tag{1}$$

The annealed-dilution network with binary couplings is specified by the two constraints [2] :

$$J_j = 0, \pm 1 \tag{2}$$

$$\sum_{j=1}^{N} J_j^2 = fN. \tag{3}$$

The parameter $f \in [0,1]$ determines the degree of dilution. For $f = 1$, the model reduces to the fully-connected network studied by Krauth and Mézard [1]. The storage capacity $\alpha_c(f)$ is defined in the thermodynamic limit $N \to \infty$, $p \to \infty$ as the largest value of $\alpha = \frac{p}{N}$ for which coupling coefficients J_j can be found that fulfill all conditions (1) (2) (3).

Following the approach of Gardner-Derrida [3], an energy function is defined on the space of coupling vectors $\boldsymbol{J}$ that counts the number of stability conditions (1) that are violated. The associated free energy $F(T) = -\frac{T}{N} < \ln Z(T) >$

ESANN'1994 proceedings - European Symposium on Artificial Neural Networks
Brussels (Belgium), 20-21-22 April 1994, D-Facto public., ISBN 2-9600049-1-4, pp. 55-60

can then be calculated using the replica method.This has been done in ref [2] using the replica symmetry ansatz and the ad hoc assumption that the vanishing of the entropy marks the critical storage capacity.Here we go one step futher and use the first-stage replica symmetry breaking ansatz [4]. This yields

$$F_{RSB}^{(1)}(T) = -T \operatorname*{Extr}_{\{q_0 q_1 m \hat{q}_0 \hat{q}_1 \hat{f}\}} G_{RSB}^{(1)}(q_0, q_1, m, \hat{q}_0, \hat{q}_1, \hat{f}; \alpha, T, f) \tag{4}$$

where

$$G_{RSB}^{(1)} = \frac{1}{2} f(1-m) q_1 \hat{q}_1 + \frac{1}{2} f m q_0 \hat{q}_0 - f \hat{f}$$
$$+ \frac{1}{m} \int_{-\infty}^{+\infty} Dz_0 \ln \int_{-\infty}^{+\infty} Dz_1 \left[1 + e^{\hat{f} - \frac{\hat{q}_1}{2}} 2 \cosh \left(\sqrt{\hat{q}_0} z_0 + \sqrt{\hat{q}_1 - \hat{q}_0} z_1 \right) \right]^m$$
$$+ \frac{\alpha}{m} \int_{-\infty}^{+\infty} Dz_0 \ln \int_{-\infty}^{+\infty} Dz_1 \left[\int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{2\pi(1-q_1)}} e^{-\frac{1}{T}\theta[K-\lambda] - \frac{(\lambda - \sqrt{q_0} z_0 - \sqrt{q_1 - q_0} z_1)^2}{2(1-q_1)}} \right]^m \tag{5}$$

The extremum yields six complicated saddle-point equations and finding a solution – different from the RS solution – turns out to be a daunting numerical task. We have concentrated on this problem for quite some time but, in spite of considerable effort, have failed to find a new solution that satisfies all equations. This situation resembles that encountered by Krauth and Mézard [3].

Trying to understand the behaviour of the free energy we studied the value of the function

$$\tilde{G}(q_1) = \operatorname*{Extr}_{\{q_0 m \hat{q}_0 \hat{q}_1 \hat{f}\}} G_{RSB}^{(1)} \tag{6}$$

on the interval $q_1 \in [0,1]$. Extensive numerical calculations for different value of f, α and T show that for high α and low T, the lowest values of $\tilde{G}(q_1)$ is systematically found at $q_1 = 1$. This result is illustrated in figure 1 for two choices of (f, α). The left figure, for $f = 0.6$, $\alpha = 1.4$, shows the function $T\tilde{G}(q_1)$ in the interval $q_1 \in [0.4, 1]$ for two values of the temperature.[1] The lower curve, at the temperature $T = 0.14$, is typical for the case when only the RS solution exists. At low values of q_1, the extremum in (6) yields $q_0 = q_1$. The value of $\tilde{G}(q_1)$ rapidly decreases with increasing q_1 until the minimum value is reached at $q_1 = 0.67$. This is the RS solution. A further increase in q_1 does not alter the value of $\tilde{G}(q_1)$. This constancy is achieved as follows. For $q_1 > 0.67$, the extremum in (6) systematically yields $m = 1$ and $q_0 = 0.67$, thereby keeping the Parisi order parameter function [4] unchanged. The upper curve, at the *lower* temperature $T = 0.05$, shows a similar behaviour at low values of q_1. The RS

[1] The extra temperature factor added to $\tilde{G}$ has the effect that the lowermost value of each curve equals minus the free energy at the corresponding temperature (see (4)).

ESANN'1994 proceedings - European Symposium on Artificial Neural Networks
Brussels (Belgium), 20-21-22 April 1994, D-Facto public., ISBN 2-9600049-1-4, pp. 55-60

solution is reached now at $q_1 = 0.81$. It is followed by a plateau along which q_0 stays close to 0.81 and m close to 1. At high values of q_1, though, the function $\tilde{G}(q_1)$ begins to slope down again and, very close to the boundary $q_1 = 1$, it plunges towards its lowest value at $q_1 = 1$. We have calculated $\tilde{G}(q_1)$ numerically up to $q_1 = 0.998$ (solid curve) and have extrapolated (dotted curve) to $q_1 = 1$. It is clear from the figure that the derivative $\tilde{G}'(q_1)$ does not vanish at $q_1 = 1$, which explains our lack of success in solving the *six* saddle-point equations for the extremum of (4). The right part of figure 1 shows the same behaviour for the non-diluted case $f = 1$, $\alpha = 1$, at the temperature $T = 0.19$ (lower curve) and at the *lower* temperature $T = 0.1$ (upper curve). Similar figures have been obtained for other values of f.

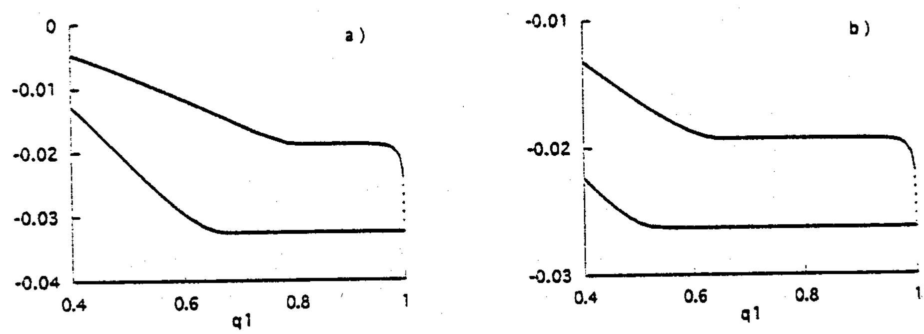

Figure 1: a) The function $T\tilde{G}(q_1)$ for $f = 0.6$, $\alpha = 1.4$ for $T = 0.14$ (lower curve) and $T = 0.05$ (upper curve). b) The function $T\tilde{G}(q_1)$ for $f = 1$, $\alpha = 1$ for $T = 0.19$ (lower curve) and $T = 0.1$ (upper curve)

As suggested by the numerical results, we have searched for an extremum of $G^{(1)}_{RSB}$ within the hyperplane $q_1 = 1$ at finite temperature. Since $q_1 = 1$ is at the boundary of the domain of q_1, its conjugate $\hat{q}_1$ must tend to infinity. The crucial point now is to notice that, as $\hat{q}_1 \to \infty$, there must be a concurrent growth of $\hat{f}$ in order to keep the dilution degree intermediate between 0 and 1. More explicitly, the combination

$$\hat{r} := m\hat{f} - \frac{m(1-m)}{2}\hat{q}_1 \tag{7}$$

has to retain a finite value. Once this is noted, it becomes straightforward to calculate the limit of $G^{(1)}_{RSB}$:

$$G^{(1)}_{RSB} \underset{\substack{q_1 \to 1 \\ \hat{q}_1 \to \infty}}{\longrightarrow} \frac{1}{m}\left\{\frac{1}{2}fm^2\hat{q}_0 q_0 - f\hat{r} + \int_{-\infty}^{+\infty} Dz_0 \ln[1 + e^{\hat{r} - \frac{m^2\hat{q}_0}{2}} 2\cosh(m\sqrt{\hat{q}_0}z_0)]\right\}$$

$$+\alpha \int_{-\infty}^{+\infty} Dz_0 \ln\left[\int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{2\pi(1-q_0)}} e^{\frac{m}{T}\theta[K-\lambda] - \frac{(\lambda - \sqrt{q_0}z)^2}{2(1-q_0)}}\right]. \tag{8}$$

Comparison with the RS expression G_{RS} shows

$$G^{(1)}_{RSB} \underset{\substack{q_1 \to 1 \\ \hat{q}_1 \to \infty}}{\longrightarrow} \frac{1}{m} G_{RS}[q_0, m^2\hat{q}_0, \hat{r}; \alpha, \frac{T}{m}, f]. \tag{9}$$

Plugging the expression (9) into (4) yields

$$F^{(1)}_{RSB}(T) = \underset{\{q_0 m \hat{q}_0 \hat{r}\}}{\text{Extr}} \left[-\frac{T}{m} G_{RS}[q_0, m^2\hat{q}_0, \hat{r}; \alpha, \frac{T}{m}, f]\right]. \tag{10}$$

The main difference with the expression for $F_{RS}(T)$ is the occurrence of the extra variable m in the Extremum operation. However, if we rewrite (10) as

$$F^{(1)}_{RSB}(T) = \underset{m}{\text{Extr}}\left\{\underset{\{q_0 \hat{q}_0 \hat{r}\}}{\text{Extr}} \left[-\frac{T}{m} G_{RS}[q_0, m^2\hat{q}_0, \hat{r}; \alpha, \frac{T}{m}, f]\right]\right\}, \tag{11}$$

we find that

$$F^{(1)}_{RSB}(T) = \underset{m}{\text{Extr}}\, F_{RS}\left(\frac{T}{m}\right). \tag{12}$$

This yields as equation for m

$$F'_{RS}\left(\frac{T}{m}\right) = 0 \tag{13}$$

which means that the RS entropy must vanish at $\frac{T}{m}$. Study of the function $F_{RS}(T)$, shows that it does not display a stationary point as long as α is smaller then a special value $\alpha_s(f)$. Hence, no RSB solution exists in this case. When $\alpha > \alpha_s(f)$, on the other hand, $F_{RS}(T)$ displays a maximum at $T_c(\alpha, f)$, which yields the solution $m = \frac{T}{T_c(\alpha, f)}$ of equation (13). Since m is constrained by $0 \leq m \leq 1$, this solution is acceptable only when $T < T_c(\alpha, f)$. The RSB free energy (12) becomes

$$F^{(1)}_{RSB}(T) = F_{RS}(T_c) \qquad (\alpha > \alpha_s(f); T < T_c(\alpha, f)) \tag{14}$$

ESANN'1994 proceedings - European Symposium on Artificial Neural Networks
Brussels (Belgium), 20-21-22 April 1994, D-Facto public., ISBN 2-9600049-1-4, pp. 55-60

As $F_{RS}(T_c)$ is a positive constant, the RSB entropy vanishes and the energy remains positive for all $T < T_c$. Thus, for $\alpha > \alpha_s(f)$, part of the stability conditions (1) always remains violated. By contrast, for $\alpha < \alpha_s(f)$, where the RS solution is the sole solution, cooling the system will bring it to its ground state which now has zero energy i.e. all patterns are safely stored. The transition point $\alpha_s(f)$, characterized by the vanishing of the RS entropy at $T = 0$, thus determines the storage capacity $\alpha_c(f)$ of the annealed-dilution network with binary couplings.

To conclude, we present in figure 2 our result for the storage capacity $\alpha_c(f)$ for the case where the stability parameter K is chosen equal to zero. Here, we have added some numerical estimate of $\alpha_c(f)$ as determined through a full enumeration, for a system of $N = 17$ neurons, of all possible coupling vectors. Choosing a set of $p = \alpha N$ random input-output pairs, we scan all possible coupling vectors to find out whether or not a vector exists that stabilizes all patterns. Repeating this process for many samples determines the probability of success which then determines the storage capacity. The overall agreement with the theoretical curve is quite good in spite of the small number of neurons for which a full enumeration of all coupling vectors can be carried out.

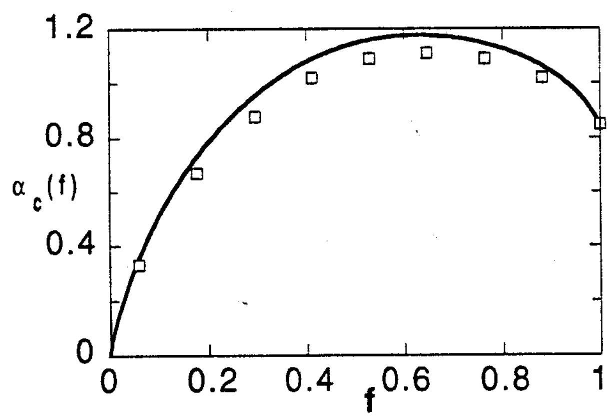

Figure 2: The storage capacity $\alpha_c(f)$ as function of the fraction of non-zero couplings for $K = 0$. The little squares are obtained from a full enumeration of coupling vectors for a network with $N = 17$.

References

1. Krauth W and Mézard M, J. Phys. (Paris) **50** (1989) 3057.

2. Bouten M , Komoda A and Serneels R, J. Phys. A **23** (1990) 2605.

ESANN'1994 proceedings - European Symposium on Artificial Neural Networks
Brussels (Belgium), 20-21-22 April 1994, D-Facto public., ISBN 2-9600049-1-4, pp. 55-60

3. Gardner E and Derrida B, J. Phys. A **21** (1988) 271.

4. Parisi G, J. Phys. A **13** (1980) 1101.