ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

Neural networks for the solution of information-distributed optimal control problems

M. Baglietto,* T. Parisini,** and R. Zoppoli*
e-mail: [giassai,thomas,rzop]@dist.unige.it

*DIST–University of Genoa, Via Opera Pia 13, 16145 Genova,Italy
**DEEI–University of Trieste, Via Valerio 10, 34175 Trieste,Italy

Abstract. The method of constraining unknown control functions to take the structures of suitable nonlinear approximators enables one to solve approximately (but at any desired degree of accuracy and without the need of too many parameters to optimize) "difficult" functional optimization problems. The method is tested on Witsenhausen's counterexample, using neural networks as nonlinear approximators. *The simulation results* show the effectiveness of the proposed method.

1. Information-distributed optimal control problems and an approximate method for solving them

There are many situations, in engineering and economic systems, where several decision makers (DMs), sharing different information patterns, cooperate to the accomplishment of a common goal. Typical examples may be encountered in communication networks, in large-scale traffic systems, in geographically distributed control systems, etc. As the DMs cooperate to the minimization of a common cost function, a *team optimal control problem* is faced.

Solving analytically such a problem under general assumptions is a practically impossible task. In [1], Ho and Chu, developing the pioneering work by Radner [2], gave sufficient conditions for solving a team problem, that is, when *i)* the team problem is LQG and *ii)* the information structure is partially nested, i.e., when any DM can reconstruct the information of the DMs the actions of which influenced its own information. Unfortunately, most of team organizations do not satisfy the aforesaid sufficient conditions. This leads us to address an approximate technique consisting in constraining the control functions to have a fixed structure (we chose feedforward neural networks). We are then able to obtain suboptimal solutions under very general conditions. Such a technique has proved to be effective in non-LQG classical optimal control and in team problems not solvable analytically (see, for instance, [3, 4, 5]). The proposed method aims at solving approximately the following functional optimization problem:

ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

Problem 1. *Given* i) *a set of N decision makers* $DM_1, \ldots, DM_N$, *each implementing a decision strategy* $u_i = \gamma_i(y_i)$, $i = 1, \ldots N$, ii) *a set of information functions* $y_i = g_i(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_N, \xi)$ *satisfying certain causality conditions [6] (*ξ *is a random vector with a known probability density function), and* iii) *a cost function* $J(u_1, \ldots, u_N, \xi)$, *find the optimal strategies* $\gamma_1^\circ, \ldots, \gamma_N^\circ$ *that minimize the expected value of* J. □

Problem 1, besides being a team problem, may be regarded as a classical centralized one if the DMs correspond to the control actions generated by a single perfect-memory controller acting at stages $0, 1, \ldots, N-1$, and if ξ is composed of the initial state and of the process and measurement random noises (the state equation is implicitly included in the information functions and the cost function). The method proposed to solve Problem 1 approximately consists in the following two steps.

1. The DMs' control strategies are given fixed structures of the form $\hat{\gamma}_i(y_i, w_i)$, where w_i is a vector of parameters to be optimized. As fixed structures, we may use nonlinear approximators, like feedforward neural networks, radial basis functions (RBFs), Jones's linear combinations of trigonometric basis functions with adaptable frequencies and phases, Breiman's sums of hinge functions with adaptable hinges, and others. These approximators benefit by powerful approximation properties, that is, *i)* they are dense in the sets of continuous functions to be approximated defined on compact domains , and *ii)* under suitable assumptions on the regularity of the functions to be approximated (denote one of them by $\varphi(x) : \mathbb{R}^n \to \mathbb{R}^m$), the number of parameters needed to achieve a given integrated square error grows only linearly with n. The latter property may not hold true for linear approximators (i.e., linear combinations of fixed basis functions), for which the number of coefficients of the linear combinations may grow exponentially with n [7].

Substitution of the functions $\hat{\gamma}_i$ and g_i into the cost J yields a new cost of the form $\hat{J}(w, \xi)$, where $w \triangleq \text{col}(w_1, \ldots, w_N)$. Then, instead of the original functional optimization problem, we have to solve the following nonlinear programming problem:

Problem 2. *Find the optimal vector* w° *that minimizes* $\mathrm{E}[\hat{J}(w, \xi)]$. □

If linear approximators are used instead of nonlinear ones, step 1 resembles the Ritz method.

2. Problem 2 can be solved by some descent algorithm. We focus our attention on gradient algorithms mainly for their simplicity. This will enable us to introduce, in a straightforward way, the concept of *stochastic approximation.* Actually, due to the generality of Problem 2, we are not able to express the average cost $\mathop{\mathrm{E}}_{\xi}\left[\hat{J}\,(w, \xi)\right]$ in explicit form. This leads us to compute the *"realization"* $\nabla_w \hat{J}\,[w(h), \xi(h)]$ and to use the following updating algorithm

$$w\,(h+1) = w\,(h) - \alpha(h) \nabla_w \hat{J}\,[\,w\,(h), \xi(h)\,]\,, \; h = 0, 1, \ldots$$

ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

where the sequence $\{\xi(h), h = 0, 1, \ldots\}$ is generated by randomly selecting ξ according to its p.d.f. Note that the step-size $\alpha(h)$ must suitably decrease to ensure (hopefully) convergence (see [8] for more details). In the numerical examples, we shall take $\alpha(h) = c_1/(c_2 + h)$.

2. Witsenhausen's counterexample

Witsenhausen's well-known counterexample [9] exhibits the essential difficulties encountered when a partially nested information structure does not hold, even if LQG assumptions are verified. Following [9], let us consider two decision makers, DM_1 and DM_2, which make use of different information functions, i.e.,

$$y_1 = x\,, \quad y_2 = x + u_1 + v \tag{1}$$

where $x \sim N(0, \sigma_x^2)$ and $v \sim N(0,1)$ are two independent Gaussian random variables. Then, we can state the following

Problem W. *Find the optimal strategies γ_1° and γ_2° that minimize the cost functional*

$$\mathop{\mathrm{E}}_{x,v}\left[J(\gamma_1, \gamma_2, x, v)\right] = \mathop{\mathrm{E}}_{x,v}\left\{k^2[x - \gamma_1(y_1)]^2 + [\gamma_1(y_1) - \gamma_2(y_2)]^2\right\}$$

□

Witsenhausen demonstrated that the best affine solution of this problem is not guaranteed to be optimal. He did so by using the functions

$$\gamma_1(y_1) = \sigma_x \ \mathrm{sgn}(y_1)\,, \quad \gamma_2(y_2) = \sigma_x \ \tanh(\sigma_x\, y_2) \tag{2}$$

which, for $k^2\sigma_x^2 = 1$ and $k \to 0$, give a lower cost than the one given by the best solution in the class of affine functions. But an optimal solution was not found. No progress in deriving an optimal solution was obtained by discretizing Problem W [10]. This was explained in [11] by demonstrating that the discretized Problem W expressed in decisional form is NP-complete. In [12], an improvement over the solutions (2) was proposed by constraining γ_1 to take on the structure $\gamma_1(y_1) = \epsilon \ \mathrm{sgn}(y_1)$ and by letting the structure of γ_2 be "free". It can be shown [12] that the cost can be expressed as an explicit function of ϵ. Denote it by $\tilde{J}(\epsilon, k, \sigma_x)$. For $k \to 0$ and $k^2\sigma_x^2 = 1$, a lower cost was obtained in [12] than the one derived by Witsenhausen. In the following, we shall consider the values of ϵ that minimize $\tilde{J}(\epsilon, k, \sigma_x)$. The corresponding DMs' strategies will be called *optimized Witsenhausen solutions.*

We want to point out that all the analytical nonlinear solutions proposed in the literature outperform the best affine solutions only in a limited portion of the plane of k^2 and σ_x^2. In Fig.1 the portion of this plane is drawn in which the cost relative to the optimized Witsenhausen solutions ($J_{W\,opt}^\circ$) is lower than that relative to the best affine ones (J_{Wa}°). Not surprisingly, such an area encompasses (for not too large values of k) the curve $\sigma_x^2 k^2 = 1$ (dashed line) considered in Witsenhausen's counterexample.

ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

Finally, let us address the following problem stated by Bansal and Başar [12]:

Problem BB. *Find the optimal strategies* γ_1^* *and* γ_2^* *that minimize the cost functional*

$$\mathop{\mathrm{E}}_{x,v}\left[J(\gamma_1,\gamma_2,x,v)\right] = \mathop{\mathrm{E}}_{x,v}\left\{k_0^2\gamma_1^2(y_1) + s_{01}\gamma_1(y_1)\,x + [x-\gamma_2(y_2)]^2\right\}$$

with information structure given by (1) and $x \sim N(0,\sigma_x^2)$ *and* $v \sim N(0,\sigma_v^2)$. □
The problem is proved to have optimal affine solutions given by

$$\gamma_1^*(y_1) = \lambda^* y_1\,,\ \gamma_2^*(y_2) = \frac{\lambda^*\sigma_x^2}{\lambda^{*2}\sigma_x^2+\sigma_v^2}y_2 \tag{3}$$

where λ^* is a root of a certain 5th-order equation.

3. Numerical results

In section 2., we have introduced Problem BB in order to test our approximate method on a problem whose optimal solutions are known. We have introduced Problem W to understand the behavior of the method in finding approximately the unknown solutions of problem W. Towards this end, we shall evaluate the cost $\mathrm{E}[\hat{J}(w_1^\circ, w_2^\circ, x, v)]$, related to the various approximate optimal strategies $\hat{\gamma}_1(y_1, w_1^\circ)$ and $\hat{\gamma}_2(y_2, w_2^\circ)$, by its empirical estimate $J_e^\circ \triangleq \frac{1}{M}\sum_{j=1}^{M}[w_1^\circ, w_2^\circ, x(j), v(j)]$. $x(j)$ and $v(j)$ are generated randomly on the basis of their probability densities, and M is a sufficiently large integer (we shall take $M = 10^5$). The costs for the optimal solutions of Problem BB (J_{BB}°), for the optimal affine solutions of Problem W (J_{Wa}°), and for the optimized Witsenhausen solutions ($J_{W opt}^\circ$) can be evaluated analytically.
a) *Comparison between optimal solutions of Problem BB and approximate optimal solutions that solve the related Problem 2.* In Fig.3A, the strategies $\hat{\gamma}_1(y_1, w_1^\circ)$ and $\hat{\gamma}_2(y_2, w_2^\circ)$ are compared with the strategies (3). In this example and in the following ones, the strategies $\hat{\gamma}_1$ and $\hat{\gamma}_2$ were implemented by using feedforward neural network with a single hidden layer with 30 sigmoidal units and a linear output unit. We obtained the costs $J_{BB}^\circ = 8.913$ and $J_e^\circ = 8.915$. The training phase is plotted in Fig. 2. As can be seen, the approximate strategies practically coincide with the optimal ones in the statistically significant ranges. This was verified for several combinations of parameters. Similar results were obtained by using RBFs and Jones's approximators.
b) *Comparison between optimized Witsenhausen solutions, optimal affine solutions, and approximate optimal solutions that solve the related Problem 2.* In Fig.3B, the comparison is made for the case $k^2 = 10$, $\sigma_x^2 = 10$, for which the best affine solution outperforms the optimized Witsenhausen one. The opposite occurs for the case $k^2 = 0.1$, $\sigma_x^2 = 10$ (see Fig.3C. In the former case, we obtained the costs $J_{Wa}^\circ = 0.909$, $J_{W opt}^\circ = 36.4$ and $J_e^\circ = 0.922$; in the latter, $J_{Wa}^\circ = 0.9090$, $J_{W opt}^\circ = 0.417$ and $J_e^\circ = 0.409$. *It is important to point out that*

ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

our approximate method was able to yield function shapes similar to the best known ones (whether they are linear or nonlinear), irrespectively of the values of the problem parameters. More precisely, as is shown in Fig.3C, the approximate strategy $\hat{\gamma}_1$ does not take exactly a step shape, but a linear component is also present (a structure of this type was proposed in [12]). The same occurs also for the shape of $\hat{\gamma}_2$. It is worth noting that, for some particular values of the parameters, the approximate optimal strategies yield better costs than the ones so far derived in the literature. This deserves a further investigation in order to understand if the optimal strategies derived in solving Problem 2 perform better than the best known ones in the whole plane k^2, σ_x^2 (provided that a sufficient number of neural units is used). As to the behaviors of the various nonlinear approximators, we may report that (apart from the request for more or fewer units) neural networks, RBFs, and Jones's approximators behaved in the same way.

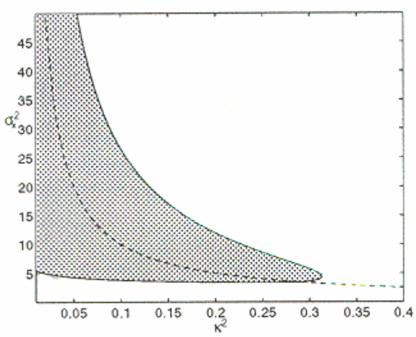

Fig. 1: In the grey zone $J^{\circ}_{W\,opt} < J^{\circ}_{W\,a}$.

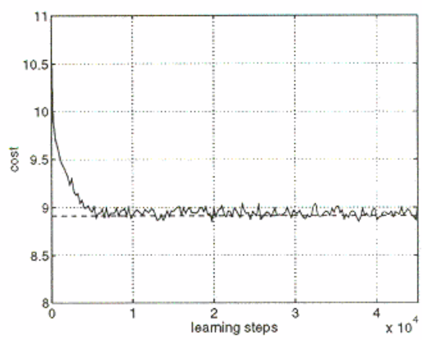

Fig. 2: Decrease in the cost.

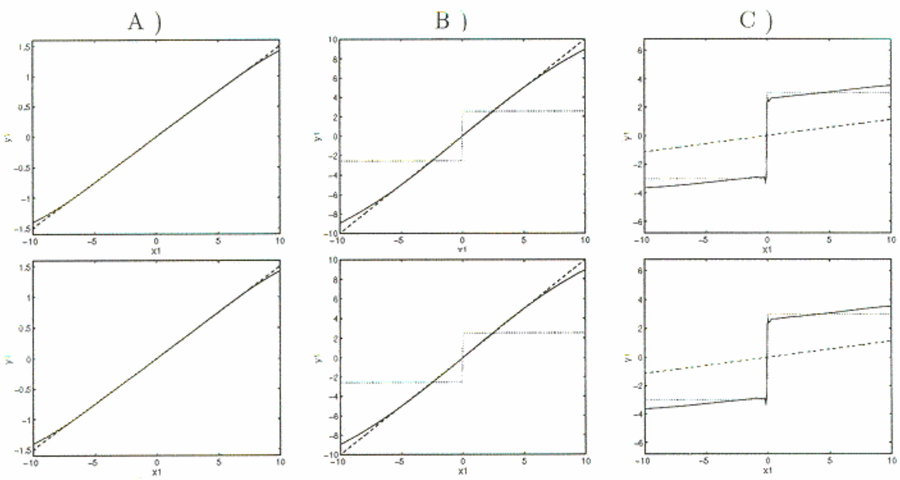

Fig. 3: **A)** Neural control functions (——) and optimal solutions (- - -) for a Problem BB with $k_0 = 10, s_{01} = -1, \sigma_x^2 = 10$, and $\sigma_w^2 = 1$. **B,C**) Neural control functions (——), best affine solutions (- - -) and optimized Witsenhausen solutions ($\cdots$) for the cases $k^2 = 10, \sigma_x^2 = 10$ and $k^2 = 0.1, \sigma_x^2 = 10$ respectively.

ESANN'1998 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 22-23-24 April 1998, D-Facto public., ISBN 2-9600049-8-1, pp. 79-84

References

[1] Y. C. Ho and K. C. Chu, "Team decision theory and information structures in optimal control problems," *IEEE Trans. on Automatic Control*, vol. AC-17, no. 1, pp. 15–28, June 1972.

[2] R. Radner, "Team decision problems," *Ann. Math. Statist.*, vol. 33, no. 3, pp. 857–881, 1962.

[3] T. Parisini and R. Zoppoli, "Team theory and neural networks for dynamic routing in traffic and communication networks," *Information and Decision Technologies*, vol. 19, pp. 1–18, 1993.

[4] T. Parisini and R. Zoppoli, "Neural networks for feedback feedforward nonlinear control systems," *IEEE Trans. on Neural Networks*, vol. 5, pp. 436–449, 1994.

[5] T. Parisini and R. Zoppoli, "Neural approximations for multistage optimal control of nonlinear stochastic systems," *IEEE Trans. on Automatic Control*, vol. 41, no. 6, pp. 889–895, 1996.

[6] M. S. Andersland, "Decoupling non-sequential stochastic control problems," *Systems and Control Letters*, vol. 16, pp. 65–69, 1991.

[7] A. R. Barron, "Universal approximation bounds for superpositions of a sigmoidal function," *IEEE Trans. on Information Theory*, vol. 39, pp. 930–945, 1993.

[8] B. T. Polyak and Ya. Z. Tsypkin, "Pseudogradient adaptation and training algorithms," *Automation and Remote Control*, vol. 12, pp. 377–397, 1973.

[9] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM J. Control*, vol. 6, no. 1, pp. 131–147, 1968.

[10] Y. C. Ho and T. S. Chang, "Another look at the nonclassical information structure problem," *IEEE Trans. on Automatic Control*, vol. AC-25, no. 3, pp. 537–540, June 1980.

[11] C. H. Papadimitriou and J. N. Tsitsiklis, "Intractable problems in control theory," *SIAM Journal on Control and Optimization*, vol. 24, no. 4, pp. 639–654, 1986.

[12] R. Bansal and Başar T, "Stochastic teams with nonclassical information revisited: When is an affine law optimal?," *IEEE Trans. on Automatic Control*, vol. AC-32, no. 6, pp. 554–559, June 1987.