ESANN'2007 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 25-27 April 2007, d-side publi., ISBN 2-930307-07-2.

Order in Complex Systems of Nonlinear Oscillators: Phase Locked Subspaces

Jan-Hendrik Schleimer and Ricaro Vigário

Helsinki University of Technology
Laboratory of Computer & Information Science
P.O. Box 5400, FI-02015 HUT - Finland

Abstract. Any order parameter quantifying the degree of organisation in a physical system can be studied in connection to source extraction algorithms. Independent component analysis (ICA) by minimising the mutual information of the sources falls into that line of thought, since it can be interpreted as searching components with low complexity. Complexity pursuit, a modification minimising Kolmogorov complexity, is a further example. In this article a specific case of order in complex networks of self-sustained oscillators is discussed, with the objective of recovering original synchronisation pattern between them. The approach is put in relation with ICA.

1 Interactions in complex systems

Synchronisation is a commonly encountered phenomenon in complex systems consisting of many interacting nonlinear oscillatory elements, each with a stable limit cycle [1]. Such dynamic systems are present in nature, *e.g.*, nervous systems [2] or chemical oscillators [3], and in technical devices like Josephson's junction [4]. Inference about the interactions in complex systems is also related to such omnipresent phenomena as self-organisation and the $\frac{1}{f}$ noise [5].

The quantification of synchronisation in complex systems from empirical measurements is accompanied with a difficulty: access to the individual oscillators is crucial otherwise spurious synchronisation will be measured [6]. In many fields these are not easily available, but one rather deals with superpositions of several elementary oscillators [7]. Below it is addressed how the original oscillators could be regained by postulating a general synchronisation structure for the system.

2 Superpositions and phase synchrony

Phase synchrony between two oscillators $u_j(t)$ and $u_k(t)$ with arbitrary phase lag can be quantified with the phase locking factor (PLF), the circular variance of the phase lag between them. Formally it is defined as the amplitude $\varrho_{jk} \in \mathbb{R}$ of the complex variable

$$\varrho_{jk}\ e^{\mathrm{i}\Psi_{jk}} = \left\langle e^{\mathrm{i}(\phi_j - \phi_k)} \right\rangle, \qquad \Longrightarrow \quad 0 \leqslant \varrho_{jk} \leqslant 1, \tag{1}$$

where ϕ_j, ϕ_k are the instantaneous phase variables of u_j and u_k. Here the expectation $\langle \cdot \rangle$ is taken over time, assuming that phase synchrony is persistent in the time interval. Also, *e.g.*, in EEG event related studies, the expectation could be taken over epochs of repeated stimulations.

Measured signals $\boldsymbol{y}(t)$ often manifest superpositions of various oscillators. If the PLF is calculated among $\boldsymbol{y}$, the result will not represent the interaction between the sources. When mixing fully phase synchronous with asynchronous sources (*cf.* example in Fig. 1a), the resulting synchrony will be: (i) observed all across the measurements and (ii) in general weaker than between the perfectly locked sources (as can be seen in Fig. 1b).

To illustrate what happens with two phase locked oscillators of the same subspace when superimposed, let us revisit the example in [6]

Ex. 1. Let $u_k(t) = U_k(t)\cos(\phi_k(t))|_{k=1,2}$, $U_j(t) > 0$ be two oscillators with a constant phase lag $\Delta\phi = \phi_1(t) - \phi_2(t)$. Consider a superposition $y_j(t) = Y_j(t)\cos(\psi_j(t)) = a_{j1}u_1(t) + a_{j2}u_2(t)|_{j=1,2}$, where the matrix (a_{jk}) is of full rank. The phase is

$$\psi_j(t) = \phi_1(t) + \arctan\left(\frac{a_{j2}U_2(t)\cos(\Delta\phi)}{a_{j1}U_1(t) + a_{j2}U_2(t)\sin(\Delta\phi)}\right),$$

from which we can find a lower bound to the change of the phase difference[1]

$$\left|\frac{d\Delta\psi}{dt}\right| \geqslant \left|\left|\frac{a_{11}a_{12}\sin\Delta\phi(\dot{U}_1U_2 - U_1\dot{U}_2)}{(a_{11}U_1 + a_{12}U_2)^2}\right| - \left|\frac{a_{21}a_{22}\sin\Delta\phi(\dot{U}_2U_1 - U_2\dot{U}_1)}{(a_{21}U_1 + a_{22}U_2)^2}\right|\right|.$$

The magnitude of the change of $\Delta\psi$ is greater than zero if $\Delta\phi \neq 0$, $\dot{U}_j \neq 0$, $\dot{U}_1/U_1 \neq \dot{U}_2/U_2$ and $\det(a_{jk}) \neq 0$. As a consequence, the phase lag of linearly transformed, locked oscillators can be non-constant over time which will reduce the phase locking factor (Eq. (1)) to a value below one, $\varrho_{12} < 1$.

The goal is to transform the observations back to the source space, reducing spurious synchronisation, in order to identify coupled oscillators therein. In [6, 7] independent component analysis (ICA, *cf.* [8] and Refs. therein) was employed to reverse the superposition process before calculating the PLF. ICA, as a specific approach to the blind source separation problem, tries to infer a multivariate system $\boldsymbol{u}(t)$ from observations $\boldsymbol{y}(t)$ generated by linear superpositions $\boldsymbol{y}(t) = \boldsymbol{M}\boldsymbol{u}(t)$. No explicite knowledge of $\boldsymbol{M}$ is summoned except that it is invertible, *i.e.*, $\exists \boldsymbol{W}^\mathsf{T} = \boldsymbol{M}^{-1}$. Also a rather generic assumption of statistical independence of the sources $\boldsymbol{u}$, is utilised. Seemingly general, the second assumptions may be inappropriate in some cases.

3 ICA, complexity pursuit and phase order

Weak interactions of self-sustained oscillators cause negligible perturbations of one another's amplitudes. Though, these can still be correlated if the phase portraits of the oscillators, as defined by the system parameters, are "similar".

[1] We drop explicite time dependence (t) for the sake of brevity.

As a result, the ICA assumption of statistically independent source signals may be violated in an ensemble of similar oscillators. The following describes how the assumptions can be modified. Our discussion starts with the classic formulation of ICA as minimising the mutual information I between the sources $\boldsymbol{u}$

$$I[\boldsymbol{u}] = \sum_j H[u_j] - H[\boldsymbol{y}] - \ln|\det \boldsymbol{W}|, \tag{2}$$

where the entropy, $H[\boldsymbol{y}]$, of the observations is not relevant in the optimisation of I w.r.t. $\boldsymbol{u}$ or $\boldsymbol{W}$. $\sum_j H[u_j]$ is the sum of the source entropies, and $-\ln|\det \boldsymbol{W}|$ a regularising term to be discussed below (see also [9]). Since the overall scaling of $\boldsymbol{u}(t)$ is not identifiable [8], a normalisation constraint for the columns of the demixing matrix, $\sum_j \boldsymbol{w}_j^\top \boldsymbol{w}_j = 1$, can be added without loss of generality.

The source entropy is a complexity measure, its maximum corresponding to complete randomness (disorder) and its minimum to a highly ordered or structured signal. Signals occurring in nature typically belong to the second type. Based on this observation, algorithms using approximations of Kolmogorov complexity to recover natural signals were devised [9, 10]. For concrete problems, other measures could be investigated, *e.g.*, any order parameter natural to the physical system. Instead of minimising the generalised notion of complexity, one can opt for maximising a specific measure that emphasises one's desired aspect of order.

We suggest that order in ensembles of coupled oscillator can be conceived as organising the multidimensional system into subspaces of oscillators with coupled dynamics. Explicitely, the assumptions are:

A1: Oscillators of the same subspace are completely phase locked;

A2: Between subspaces there is no Phase locking.

To measure *A1* and *A2* for a system of source oscillators u_k, let us define the following quantity.

Def. 1. The order of a system in terms of phase locked subspaces can be quantified by

$$P = -\sum_{j,k} \varrho_{jk} \ln \varrho_{jk}, \qquad \text{with } \varrho_{jk} \text{ as defined in Eq. (1). Thus } P \geqslant 0.$$

P optimised w.r.t. the projection $\boldsymbol{W}$ leads to the sources most in agreement with the assumptions.

The inference of synchronous subspaces based on P gives rise to degenerate solutions, where several columns of $\boldsymbol{W}$ converge to the same vector, with trivially synchronous projections. Substituting P, instead of $\sum_j H[u_j]$, into Eq. (2) we get

$$C = -\sum_{j,k} \varrho_{jk} \ln \varrho_{jk} - \ln|\det \boldsymbol{W}|. \tag{3}$$

The term $-\ln|\det \boldsymbol{W}|$ basically forces $\boldsymbol{W}$ to be more orthogonal, preventing the degeneracy. In the sequel its properties are discussed and it is argued to be more suitable for subspace analysis than the strict orthogonalisation used in many ICA algorithms.

ESANN'2007 proceedings - European Symposium on Artificial Neural Networks
Bruges (Belgium), 25-27 April 2007, d-side publi., ISBN 2-930307-07-2.

3.1 Phase ordering flow near Stiefel-Graßmann manifolds

Recall the definition, *cf.* [11], here only for square matrices

Def. 2. The Stiefel and Graßmann manifolds, $\mathcal{S}, \mathcal{G} \subset \mathbb{R}^{m \times m}$, are defined such that its elements $\boldsymbol{W} \in \mathcal{S} \cup \mathcal{G}$ suffice orthonormality, $\boldsymbol{W}^\mathsf{T}\boldsymbol{W} = \boldsymbol{I}$, and (i) for $\boldsymbol{W} \in \mathcal{S} : C(\boldsymbol{W}\boldsymbol{R}) \neq C(\boldsymbol{W})$ or (ii) for $\boldsymbol{W} \in \mathcal{G} : C(\boldsymbol{W}\boldsymbol{R}) = C(\boldsymbol{W})$ for a rotation matrix $\boldsymbol{R} \neq \boldsymbol{I}$. The function $C : \mathcal{S} \to \mathbb{R}$ or $C : \mathcal{G} \to \mathbb{R}$ can be, *e.g.*, an associated cost function.

Remark. For $\mathcal{G}$ the cost is invariant of the choice of basis, *i.e.*, the basis can only be fixed up to arbitrary rotations.

Let us consider the observed measurements $\boldsymbol{y}(t)$ to be pre-whitened, $\langle \boldsymbol{y}\boldsymbol{y}^\mathsf{T} \rangle = \boldsymbol{I}$, *i.e.*, we discard a-priori all correlations in the data. This can be done by a projection onto the principal directions and a scaling to unit variance. Whitening also implies that correlations between sources are determined by the projection matrix, $\langle \boldsymbol{u}\boldsymbol{u}^\mathsf{T} \rangle = \boldsymbol{W}^\mathsf{T}\boldsymbol{W}$.

ICA algorithms are typically restricted to the Stiefel manifold [11], as a consequence of which the sources are uncorrelated. This reduces the number of free parameters for the optimisation. While this is no hindrance in ICA, where sources are even assumed statistically independent, it may come as a restriction in subspace analysis, *e.g.*, it implies that the phase shift within the subspace is $\frac{\pi}{2}$. Not strictly enforcing this constraint opens the possibility for some sources to be correlated. The algorithm may then converge to the true source, and the postulated phase locking structure.

Under certain conditions, the last term in Eq. (3) evaluates how close $\boldsymbol{W}$ is to the Stiefel-Graßmann manifold. This is seen by comparing it to the term $\|\boldsymbol{W}^\mathsf{T}\boldsymbol{W} - \boldsymbol{I}\|_F^2$, which is an intuitive measure of how close $\boldsymbol{W}$ is to orthogonality.

Lemma 1. *For a matrix $\boldsymbol{W} \in \mathbb{R}^{m \times m}$ with normalised columns, $\forall j : \|\boldsymbol{w}_j\|_2 = 1$, it holds that $-\ln|\det \boldsymbol{W}| \geqslant 0$.*

This follows if the determinant is regarded as the volume of an m dimensional parallelepiped of side length one (Hadamard's inequality).

Lemma 2. *For a square matrix $\boldsymbol{W}$ near the Stiefel-Graßmann manifold, i.e., $\|\boldsymbol{W}^\mathsf{T}\boldsymbol{W} - \boldsymbol{I}\|_F \ll 1$, the two quantities $\|\boldsymbol{W}^\mathsf{T}\boldsymbol{W} - \boldsymbol{I}\|_F^2$ and $-\ln|\det \boldsymbol{W}|$ have the same minimisers for which they obtain zero.*

Proof. Let $\boldsymbol{\Xi} = \boldsymbol{W}^\mathsf{T}\boldsymbol{W}$. Using some properties of the determinant function we have $\ln|\det \boldsymbol{W}| = \frac{1}{2}\ln|\det \boldsymbol{\Xi}| = \frac{1}{2}\operatorname{tr}(\ln \boldsymbol{\Xi})$. The (1,1)-Padé approximation of the logarithm, which holds for $\|\boldsymbol{\Xi} - \boldsymbol{I}\| \ll 1$, can be applied: $\ln \boldsymbol{\Xi} \approx -2(\boldsymbol{I} - \boldsymbol{\Xi})(\boldsymbol{I} + \boldsymbol{\Xi})^{-1}$. So we have

$$-\ln|\det \boldsymbol{W}| \approx \operatorname{tr}((\boldsymbol{I} - \boldsymbol{\Xi})(\boldsymbol{I} + \boldsymbol{\Xi})^{-1}) \geqslant 0.$$

The lower bound of zero again follows from the restriction of normalised column vectors, so that the diagonal elements of $\boldsymbol{\Xi}$ are ones. Hence, $-\ln|\det \boldsymbol{W}|$ is minimised by $\boldsymbol{\Xi} = \boldsymbol{I}$ as is $\|\boldsymbol{\Xi} - \boldsymbol{I}\|_F^2 = \operatorname{tr}((\boldsymbol{\Xi} - \boldsymbol{I})(\boldsymbol{\Xi} - \boldsymbol{I})^\mathsf{T})$. □

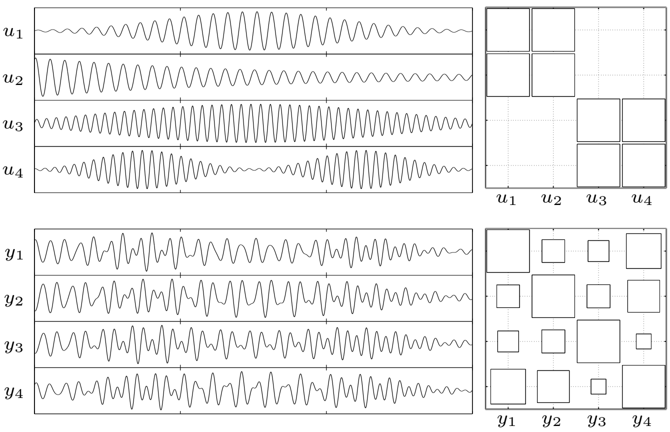

Fig. 1: (a) Oscillators $\boldsymbol{u}$ with distinct amplitudes, grouped in synchronous subspaces. PLFs are depicted as a Hinton diagram. ($P = 0.08$)

(b) Mixtures $\boldsymbol{y}(t)$ with an overall spuriously phase locking. ($P = 3.66$)

Remark. This gives us a simple intuition of $-\ln|\det \boldsymbol{W}|$ in the vicinity of Stiefel-Graßmann manifold as a force dragging a departed matrix back towards the manifold. Yet, it does not strictly enforce orthogonality, leaving the possibility for some correlations among members of a subspace. Its is also possible to introduce a hyperparameter λ to the cost $C = P - \lambda \ln|\det|\boldsymbol{W}|$, to control the trade-off between the two terms. We can also use $\|\boldsymbol{W}^{\mathsf{T}}\boldsymbol{W} - \boldsymbol{I}\|_F$ which is not restricted to square matrices, a necessary step if less sources than observations are to be extracted.

From Ex. 1 we generalise the following formal statement.

Conjecture 1. *If any two oscillators u_j and u_k belonging to the same phase locked subspace have a phase lag $\Delta\phi_{jk} > 0$, non-constant amplitude dynamics, $\dot{U}_j \neq 0$ and $\dot{U}_j/U_j \neq \dot{U}_k/U_k$, then the corresponding projection is near the Stiefel manifold and thus the rotation can be fixed.*

3.2 Gradient based optimisation

In cases where the sources agree with *A1* and *A2* they can be extracted by the gradient flow $\frac{d\boldsymbol{W}}{d\tau} = -\eta\frac{dC}{d\boldsymbol{W}} \propto \lambda\boldsymbol{W}^{-\mathsf{T}} - \frac{\partial P}{\partial \boldsymbol{W}}$. The rotation can be fixed if conjecture 1 holds. The gradient of P w.r.t. the elements of $\boldsymbol{W}$ is

$$\frac{\partial P}{\partial w_{ij}} = -4\sum_{k=1}^{m}(\ln \varrho_{jk} + 1)\left\langle \sin(\Psi_{jk} - \Delta\phi_{jk})\frac{Y_i \sin(\psi_i - \phi_j)}{U_j}\right\rangle. \quad (4)$$

For example, the four oscillators depicted in Fig. 1a fulfilling the assumptions ($P = 0.08$), but are correlated in the subspace making it more difficult for ICA. If superimposed, this structure is lost ($P = 3.66$), as is the temporal structure of the oscillator dynamics, Fig. 1b. Based on assumptions *A1* and *A2*, and with the help of the gradient flow above, the minimum phase entropy state can be recovered, together with the correct sources up to permutation, sign and amplitude scaling. Since the cost C is a nonlinear function of $\boldsymbol{W}$, the algorithm is sensitive to local minima, and to the choice of the hyperparameters η and λ.

4 Discussion

Two methods to cluster measurements directly into synchronous subspaces without reference to the superposition problem were developed recently [12, 13]. The first is a meanfield approach derived from Kuramoto's equations [3]. The second can be interpreted as spectral clustering with the phase synchronisation matrix $(\varrho_{jk})_{j,k=1..m}$ as a similarity measure. Our quantity P can be seen as a global measure of how structured this matrix is in the sense of phase locked subspaces. In a toy example, it lead to the recovery of the original sources. It is important to realise that the phase locking factor is evaluated in the source space, among the projected signals, not between measurements.

We expect that the extraction of phase synchronous subspaces by linear projections with the assumptions *A1&A2* may be of relevance to problems in neuroscience involving EEG/MEG or local field potential measurements if communication between neural units is under investigation. In that case real projection matrices as opposed to complex ones can easily be interpreted as field patterns.

The segregation of subspaces by assumptions *A1&A2* focus on synchronisation properties only. Other criterion based on amplitude statistics have been proposed to foster the projections into subspaces [14].

References

[1] A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization – A universal concept in nonlinear sciences.* Cambridge University Press, UK, 2001.

[2] J.-P. Lachaux, E. Rodriguez, J. Martinerie, and F. J. Varela. Measuring phase synchrony in brain signals. *Human Brain Mapping*, 8(4):194–208, 1999.

[3] Y. Kuramoto. *Chemical Oscillations, Waves and Turbulences.* Springer Berlin, 1984.

[4] B. D. Josephson. The discovery of tunnelling supercurrents. *Rev. Mod. Phys.*, 46(2):251–254, Apr 1974.

[5] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. *Phys. Rev. A*, 38(1):364–374, Jul 1988.

[6] F. C. Meinecke, A. Ziehe, J. Kurths, and K.-R. Müller. Measuring Phase Synchronization of Superimposed Signals. *Phys. Rev. Letters*, 94(8), 2005.

[7] B. Hong, S. Acharya, S. Gao, and N. V. Thakor. Transient phase synchrony of independent cognitive components underlying scalp EEG. In *Proc. of IEEE Engin. in Med. & Biol.*, 2005.

[8] A. Hyvärinen, J. Karhunen, and E. Oja. *Independent Component Analysis.* 2001.

[9] P. Pajunen. Blind source separation of natural signals based on approximate complexity minimization. In *Proceedings of ICA Conference '99*, pages 267–270, Jan. 1999.

[10] A. Hyvärinen. Complexity pursuit: Separating interesting components from time series. *Neural Computation*, 13:883–896, 2001.

[11] A. Edelman, T. A. Arias, and S. T. Smith. The geometry of algorithms with orthogonality constraints. *SIAM Journal on Matrix Analysis and Applications*, 20(2):303–353, 1998.

[12] C. Allefeld and J. Kurths. An approach to multivariate phase synchronization analysis and its application to event-related potentials. *Int. J. Bif. & Chaos*, 14(2):417–426, 2004.

[13] C. Allefeld, M. Müller, and J. Kurths. Eigenvalue decomposition as a generalized synchronization cluster analysis. *Int J. Bifurcation & Chaos (NDES '05)*, 2007.

[14] A. Hyvärinen, P. O. Hoyer, and Mika Inki. Topographic independent component analysis. *Neural Computation*, 13(7):1527–1558, 2000.