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Bifurcation analysis for a discrete-time Hopfield neural network of two neurons with two delays

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Abstract. In this paper, a bifurcation analysis is undertaken for a discrete-time Hopfield neural network of two neurons with two different delays and self-connections. Conditions ensuring the asymptotic stability of the null solution are found, with respect to two characteristic parameters of the system. It is shown that for certain values of these parameters, fold or Neimark-Sacker bifurcations occur, but codimension 2 (fold-Neimark-Sacker, double Neimark Sacker and resonance 1:1) bifurcations may also be present. The direction and the stability of the Neimark-Sacker bifurcations are investigated by applying the center manifold theorem and the normal form theory.

1 Introduction

Hopfield neural networks have been first considered in [1] and they have received much attention ever since because of their applicability in problems of optimization, signal processing, image processing, solving nonlinear algebraic equation, pattern recognition, associative memories and so on. Hopfield neural networks are modeled by continuous or discrete dynamical systems with or without delays.

The analysis of the dynamics of neural networks focuses on two directions: establishing stability properties and discovering periodic oscillations or bifurcations and chaotic behavior. The presence of periodic or quasi-periodic oscillations is of fundamental importance in biological and artificial systems, as they are associated with central pattern generators [2].

The dynamics of continuous-time Hopfield neural networks have been thoroughly analyzed during the past two decades, while their discrete-time counterparts have only been in the spotlight since 2000. We refer to [3, 4] for the study of exponential stability properties and of the existence of periodic solutions of discrete-time Hopfield neural networks with delays. In [5, 6, 7], a bifurcation analysis of two dimensional discrete neural networks without delays has been undertaken. In [8, 9], the bifurcation phenomena have been studied, for the case of two and n-dimensional discrete neural network model with multi-delays obtained by applying the Euler method to a continuous-time Hopfield neural network with no self-connections.

In [10], a bifurcation analysis for discrete-time Hopfield neural networks of two neurons with a single delay and self-connections has been presented, revealing the existence of fold, Neimark-Sacker as well as codimension 2 bifurcations (fold-Neimark-Sacker, double Neimark Sacker and resonance 1:1).

In this paper, we extend the results from [10] to the case of a discrete-time Hopfield neural network of two neurons with two different delays and self-connections defined by:

$$\begin{cases} x_{n+1} = ax_n + T_{11}g_1(x_{n-k_1}) + T_{12}g_2(y_{n-k_2}) \\ y_{n+1} = ay_n + T_{21}g_1(x_{n-k_1}) + T_{22}g_2(y_{n-k_2}) \end{cases} \qquad \forall n \geq \max(k_1, k_2) \qquad (1)$$

In this system $a \in (0,1)$ is the internal decay of the neurons, $T = (T_{ij})_{2\times 2}$ is the interconnection matrix, $g_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ represent the neuron input-output activations and $k_1, k_2 \in \mathbb{N}$ not both even, $k_1 \neq k_2$ represent the delays. The activation functions g_i are of class C^3 in a neighborhood of 0 and $g_i(0) = 0$.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ the function given by $g(x, y) = (g_1(x), g_2(y))^T$ and $B = TDg(0) = \begin{pmatrix} T_{11}g_1'(0) & T_{12}g_2'(0) \\ T_{21}g_1'(0) & T_{22}g_2'(0) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Throughout this paper, we will consider that $b_{11} = b_{22} = \beta$ and we denote $\delta = \det(B) = b_{11}b_{22} - b_{12}b_{21}$. For the proofs of the main results, see [11].

2 Stability and bifurcation analysis

We transform system (1) into the following system of $k_1 + k_2 + 2$ equations without delays:

$$\begin{cases} x_{n+1}^{(0)} = ax_n^{(0)} + T_{11}g_1(x_n^{(k_1)}) + T_{12}g_2(y_n^{(k_2)}) \\ x_{n+1}^{(j)} = x_n^{(j-1)} \qquad \forall j = \overline{1, k_1} \\ y_{n+1}^{(0)} = ay_n^{(0)} + T_{21}g_1(x_n^{(k_1)}) + T_{22}g_2(y_n^{(k_2)}) \\ y_{n+1}^{(j)} = y_n^{(j-1)} \qquad \forall j = \overline{1, k_2} \end{cases} \qquad \forall n \in \mathbb{N} \qquad (2)$$

where $x^{(j)} \in \mathbb{R}$, $j = \overline{0, k_1}$ and $y^{(j)} \in \mathbb{R}$, $j = \overline{0, k_2}$. When studying the stability of the steady state $\bar{0} \in \mathbb{R}^{k_1+k_2+2}$, we obtain the following characteristic polynomial:

$$char(z) = [z^{k_1}(z - a) - b_{11}][z^{k_2}(z - a) - b_{22}] - b_{12}b_{21}$$

As $b_{11} = b_{22} = \beta$ and $b_{11}b_{22} - b_{12}b_{21} = \delta$, the characteristic equation becomes:

$$z^{k_1+k_2}(z - a)^2 - \beta z^{k_2}(z - a) - \beta z^{k_1}(z - a) + \delta = 0$$

We will find the set of the parameters $(\beta, \delta) \in \mathbb{R}^2$ for which the null solution of (1) is asymptotically stable. Then, a bifurcation analysis will be undertaken along the boundary of this set, using the techniques from [12]. It will be shown that system (1) exhibits Neimark-Sacker and Fold bifurcations at the origin for certain values of the parameters (β, δ) and codimension 2 bifurcations (Fold-Neimark Sacker, strong 1 : 1 resonance, double Neimark Sacker) may also be present. We underline that, in this paper, only the bifurcations causing the loss of asymptotic stability of the null solution will be analyzed, due to their practical importance (for example, the existence of asymptotically stable closed invariant curves in the case of Neimark-Sacker bifurcations).

In the followings, a list of notations will be introduced and some mathematical results will be presented, which can be proved using basic mathematical tools:

- $m = \frac{1}{2}(k_1 + k_2)$ and $l = \frac{1}{2}|k_1 - k_2|$; remark: $\frac{1}{2} \leq l \leq m$;
- $S = \{\phi_0 = 0, \phi_1, \phi_2, ..., \phi_{[m]+1}\}$ the set of all solutions of the equation $\sin(m+1)\phi - a \sin m\phi = 0$ from the interval $[0, \pi]$;
- $\psi_1 = \frac{\pi}{2l}$ and $\theta_1 = \min(\phi_1, \psi_1)$;
- the function $c : [0, \pi] \to \mathbb{R}$, $c(\theta) = \cos(m+1)\theta - a \cos m\theta$;
- the strictly decreasing function $h : [0, \theta_1) \to \mathbb{R}$, $h(\theta) = c(\theta) \sec(l\theta)$;
- $\alpha = \lim\limits_{\theta \to \theta_1} h(\theta) = \begin{cases} c(\phi_1) \sec(l\phi_1) < 0 & \text{if } \phi_1 < \psi_1 \\ -\infty & \text{if } \phi_1 \geq \psi_1 \end{cases}$
- $h^{-1} : (\alpha, 1-a] \to [0, \theta_1)$ the inverse of the function h;
- the function $U : (\alpha, 1-a] \to (0, \infty)$, $U(\beta) = 1 + a^2 - 2a \cos(h^{-1}(\beta))$; remark: U is strictly decreasing on the interval $(\alpha, 1-a]$;
- the functions $\lambda_j : \mathbb{R} \to \mathbb{R}$, $\lambda_j(\beta) = 2c(\phi_j) \cos(l\phi_j)\beta - c(\phi_j)^2$;
- the function $L : (\alpha, 1-a] \to \mathbb{R}$, $L(\beta) = \max(\lambda_j(\beta)/j \in \{0, 1, ..., [m]+1\})$;
- β_{ij} the solution of the equation $\lambda_i(\beta) = \lambda_j(\beta)$, $i \neq j$;
- $\beta_0 = \max(\beta_{0j}/j \in \{1, 2, ..., [m]+1\}, \beta_{0j} < 0)$;
- remark: $L(\beta) = \lambda_0(\beta) = 2(1-a)\beta - (1-a)^2$ for any $\beta \in [\beta_0, 1-a]$;
- if the equation $U(\beta) = L(\beta)$ has some roots in the interval (α, β_0), then β_1 is the largest of these roots; otherwise, $\beta_1 = \alpha$.

Proposition 1. *The null solution of (1) is asymptotically stable if and only if*

$$\beta_1 < \beta < 1 - a \quad \textit{and} \quad L(\beta) < \delta < U(\beta).$$

The following bifurcation phenomena causing the loss of asymptotical stability of the null solution of (1) take place:

i. Let be $\beta \in (\beta_1, 1-a)$. When $\delta = U(\beta)$, system (1) has a Neimark-Sacker bifurcation at the origin. That is, system (1) has a unique closed invariant curve bifurcating from the origin near $\delta = U(\beta)$.

ii. Let be $\beta \in (\beta_1, \beta_0)$ such that the function L is differentiable at β. When $\delta = L(\beta)$, system (1) has a Neimark-Sacker bifurcation at the origin.

iii. Let be $\beta \in (\beta_0, 1-a)$. When $\delta = L(\beta) = 2(1-a)\beta - (1-a)^2$ system (1) has a Fold bifurcation at the origin.

iv. For $\beta = (1-a)$ and $\delta = (1-a)^2$, the system (1) has a strong $1:1$ resonant bifurcation at the origin.

v. For $\beta = \beta_0$ and $\delta = L(\beta_0) = 2(1-a)\beta_0 - (1-a)^2$, the system (1) has a Fold-Neimark-Sacker bifurcation at the origin.

vi. For $\beta = \beta_1$ and $\delta = U(\beta_1)$, the system (1) has a double Neimark-Sacker bifurcation at the origin.

vii. If there exists $\beta^\star \in (\beta_1, \beta_0)$ such that the function L is not differentiable at $\beta^\star$, then for $\beta = \beta^\star$ and $\delta = L(\beta^\star)$, the system (1) has a double Neimark-Sacker bifurcation at the origin.

3 Direction and stability of Neimark-Sacker bifurcations

Let be the function $F : \mathbb{R}^{k_1+k_2+2} \to \mathbb{R}^{k_1+k_2+2}$ given by the right hand side of system (2). Let be the operators $\hat{A} = DF(0)$, $\hat{B} = D^2F(0)$ and $\hat{C} = D^3F(0)$.

In the cases *i.* and *ii.* from Proposition 1, Neimark-Sacker bifurcations occur at the origin for systems (1) and (2):

Case *i.* $\beta \in (\beta_1, 1-a)$ and $\delta = U(\beta)$. In this case, matrix $\hat{A}$ has a simple pair of eigenvalues $z = e^{\pm i\theta}$, with $\theta = h^{-1}(\beta)$, on the unit circle.

Case *ii.* $\beta \in (\beta_1, \beta_0)$ such that the function L is differentiable at β and $\delta = L(\beta)$. Let be $j \in \{1, 2, ..., [m]+1\}$ such that $L(\beta) = \lambda_j(\beta)$. In this case, matrix $\hat{A}$ has a simple pair of eigenvalues $z = e^{\pm i\phi_j}$.

In both cases, the restriction of system (2) to its two dimensional center manifold at the critical parameter values can be transformed into the normal form written in complex coordinates [12]:

$$w \mapsto zw(1 + \frac{1}{2}d|w|^2) + O(|w|^4), \qquad w \in \mathbb{C}$$

where

$$d = \bar{z}\langle p, \hat{C}(q,q,\bar{q}) + 2\hat{B}(q, (I-\hat{A})^{-1}\hat{B}(q,\bar{q})) + \hat{B}(\bar{q}, (z^2 I - \hat{A})^{-1}\hat{B}(q,q))\rangle$$

where $\hat{A}q = zq$, $\hat{A}^T p = \bar{z}p$ and $\langle p, q\rangle = 1$ (with $\langle p, q\rangle = \bar{p}^T q$)

Direct computations provide the following result:

Proposition 2. *The vectors q and p of $\mathbb{C}^{k_1+k_2+2}$ which verify*

$$\hat{A}q = zq \quad ; \quad \hat{A}^T p = \bar{z}p \quad ; \quad \langle p, q\rangle = 1$$

are given by:
$q = (z^{k_1}q_1, z^{k_1-1}q_1, ..., zq_1, q_1, z^{k_2}q_2, z^{k_2-1}q_2, ..., zq_2, q_2)^T$
$p = (p_1, (\bar{z}-a)p_1, \bar{z}(\bar{z}-a)p_1, ..., \bar{z}^{k_1-1}(\bar{z}-a)p_1, p_2, (\bar{z}-a)p_2, \bar{z}(\bar{z}-a)p_2, ...$
$..., \bar{z}^{k_2-1}(\bar{z}-a)p_2)^T$
where $q_1 = z^{k_2}(z-a) - \beta$; $\quad q_2 = b_{21}$; $\quad \bar{p}_1 = \frac{1}{char'(z)}$; $\quad \bar{p}_2 = \frac{z^{k_1}(z-a)-\beta}{b_{21}\,char'(z)}$

Proposition 3 (see [12])**.** *The direction and stability of the Neimark-Sacker bifurcation is determined by the sign of $Re(d)$. If $Re(d) < 0$ then the bifurcation is supercritical, i.e. the closed invariant curve bifurcating from the origin is asymptotically stable. If $Re(d) > 0$, the bifurcation is subcritical, i.e. the closed invariant curve bifurcating from the origin is unstable.*

4 Example

Consider the following delayed discrete-time Hopfield neural network

$$\begin{cases} x_{n+1} = ax_n + \beta \tanh(x_{n-k_1}) - \sin(y_{n-k_2}) \\ y_{n+1} = ay_n + \gamma \tanh(x_{n-k_1}) + \beta \sin(y_{n-k_2}) \end{cases} \quad \forall n \geq \max(k_1, k_2) \qquad (3)$$

Let be $a = 0.5$ and the delays $k_1 = 4$ and $k_2 = 11$. Therefore, $m = 7.5$ and $l = 3.5$ and we have $\delta = \beta^2 + \gamma$. Using Mathematica 5.0, we compute

$S = \{0, 0.3342, 0.6832, 1.0472, 1.4207, 1.7995, 2.1813, 2.5649, 2.9492\}$ (rad);

$\psi_1 = \phi_1 = 0.3342$ (rad), $\alpha = -1.41466$, $\beta_0 = -0.0386$, $\beta_1 = -0.7251$;

and we obtain the stability domain in the (β, δ)-plain represented in Fig. 1.

For $\beta = \frac{1}{4}$, we have $L(\beta) = \lambda_0(\beta) = 0$ and $U(\beta) = 0.2567$. The origin is asymptotically stable if and only if $\gamma \in (-0.0625, 0.1942)$. At $\gamma = \gamma^\star = 0.1942$ a supercritical Neimark-Sacker bifurcation occurs as $Re(d) = -0.0441 < 0$ (see Figs. 2, 3).

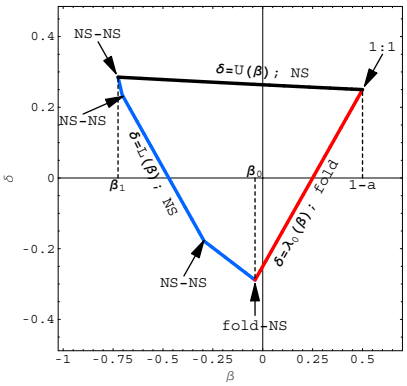

Fig. 1: The stability domain in the (β, δ)-plain for the null solution of (3)

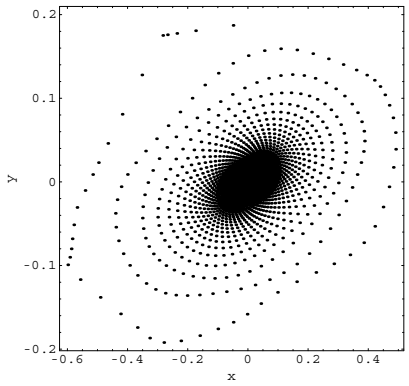

Fig. 2: For $\gamma = 0.19$, the trajectory of (3) with the initial conditions $(x_0, y_0) = (0.2, 0.2)$ converges to the asymptotically stable null solution.

5 Conclusions

The results presented in this paper complete the bifurcation results obtained for discrete-time Hopfield neural networks with a single delay and self-connections presented in [10] and for networks with two delays and no self-connections presented in [9], with new results concerning the case of neural networks with two different delays and self-connections, revealing some resemblances and some differences, as well. An extension of this analysis to the case of more general discrete-time neural networks with multi-delays and self-connections may constitute a direction for future research.

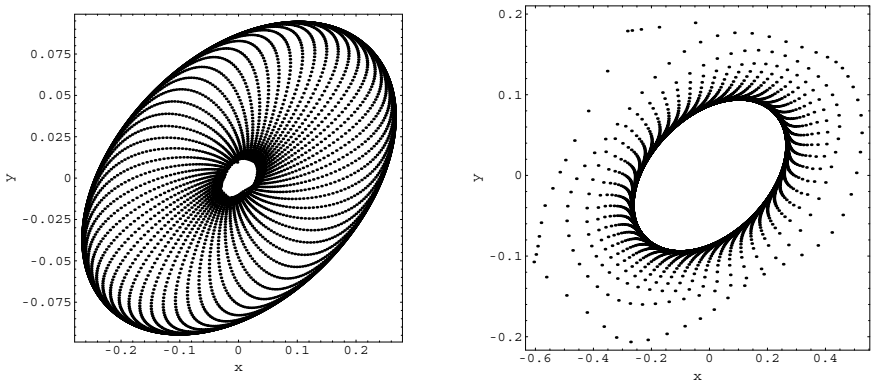

Fig. 3: For $\gamma = 0.2$, the null solution of (3) is unstable and an asymptotically stable closed invariant curve is present. The trajectories of (3) with the initial conditions $(x_0, y_0) = (0.01, 0.01)$ (left) and $(x_0, y_0) = (0.2, 0.2)$ (right) converge to the asymptotically stable invariant curve.

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